§ 4.2 Functional Limits

Our goal in this section is to make precise the meaning of limits of functions: $\lim_{x \to c} f(x) = L$.

Def Let $f: A \to \mathbb{R}$ be given, let $c \in \mathbb{R}$ be a limit point of A (not necessarily an element of A). We say the function limit $\lim_{x \to c} f(x)$ exists if there exists $L \in \mathbb{R}$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x \in A$ and $0 < |x - c| < \delta$, it follows that $|f(x) - L| < \varepsilon$. We say $\lim_{x \to c} f(x) = L$ when this happens. If no such L exists, we say $\lim_{x \to c} f(x)$ does not exist.

This can be restated as "for every ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ of c such that $x \in V_\delta(c) \cap A \setminus \{c\}$ implies $f(x) \in V_\varepsilon(L)$." (This is called the topological version of the definition).

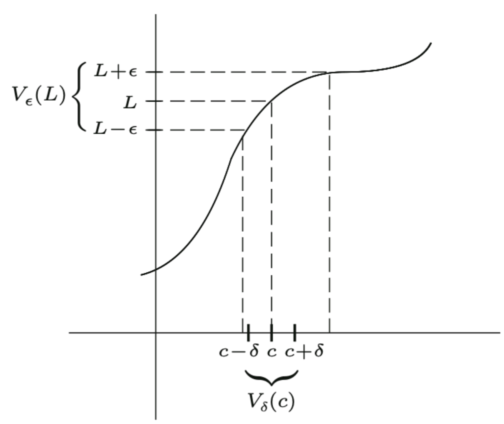

Figure 4.4: Definition of Functional Limit.

https://www.desmos.com/calculator/iejhw8zhqd

Example Give an ε - δ proof showing that if $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 2x+1 & x \neq 3 \\ 0 & x = 3 \end{cases}$$

then $\lim_{x \to 3} f(x) = 7$.

Proof Let $\varepsilon > 0$. Define $\delta = \underline{\varepsilon/2}$. Suppose $x \in \mathbb{R}$ and $0 < |x-3| < \delta$. Then

$$\begin{aligned} |f(x) - 7| &= |2x+1-7| \\ &= 2|x-3| \\ &< 2\delta \\ &= \varepsilon. \end{aligned}$$

Notice δ can depend on ε (but it should not depend on x).

Example Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

Give an ε-δ proof showing $\lim_{x \to 4} f(x) = 16$.

Proof Let $\varepsilon > 0$. Define $\delta = \underline{\min\{1, \varepsilon/9\}}$.

Suppose $x \in \mathbb{R}$ and $0 < |x-4| < \delta$. Then

$$|f(x) - 16| = |x-4||x+4|$$
$$< \delta |x+4|.$$

Observe that

$$|x+4| = |x-4+8|$$
$$\leq |x-4| + 8$$
$$< \delta + 8$$
$$\leq 9$$

where the last inequality is because $\delta \leq 1$. Therefore

$$|f(x) - 16| < 9\delta \leq \varepsilon.$$

Problem 1. For each example below, give an ϵ-δ proof that $\lim_{x\to c} f(x) = L$.

a. $f(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$, $c = 0$, $L = 0$

b. $f(x) = x^2$, $c = -3$, $L = 9$

(a) Let $\varepsilon > 0$ and define $\delta = \underline{\varepsilon^{1/2}}$. Suppose $x \in \mathbb{R}$ and $0 < |x| < \delta$. Then

$$\begin{aligned} |f(x) - L| &= \left|x^2 \sin\left(\tfrac{1}{x}\right)\right| \\ &\leq |x|^2 \\ &< \delta^2 \\ &= \varepsilon. \end{aligned}$$

(b) Let $\varepsilon > 0$ and define $\delta = \underline{\min\{1, \varepsilon/7\}}$. Suppose $x \in \mathbb{R}$ and $0 < |x+3| < \delta$

$$\begin{aligned} |f(x) - L| &= |x^2 - 9| \\ &= |x-3||x+3| \\ &\leq (|x|+3)|x+3| \\ &< (|x|+3)\,\delta \end{aligned}$$

Observe that

$$|x| = |x+3-3| \leq |x+3| + 3 < \delta + 3 \leq 4$$

and so

$$\begin{aligned} |f(x) - L| &< (|x|+3)\,\delta \\ &< 7\delta \\ &\leq \varepsilon \end{aligned}$$

Theorem Let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of A. Then the following are equivalent:

(1) $\lim_{x \to c} f(x) = L$

(2) for every sequence $(x_n) \subseteq A \setminus \{c\}$ such that $x_n \to c$, it is the case that $f(x_n) \to L$.

Proof ($\Rightarrow$) Let $(x_n) \subseteq A \setminus \{c\}$ be a sequence such that $x_n \to c$. We must show $f(x_n) \to L$.

Let $\varepsilon > 0$. Since $\lim_{x \to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in A$ and $0 < |x - c| < \delta$. Since $x_n \to c$, there exists $N \in \mathbb{N}$ such that $|x_n - c| < \delta$ when $n \geq N$. Suppose $n \geq N$. Then $x_n \in A$ and $0 < |x_n - c| < \delta$, which implies $|f(x_n) - L| < \varepsilon$.

($\Leftarrow$) Suppose by way of contradiction that $\lim_{x \to c} f(x) \neq L$.

Then there exists $\varepsilon_0 > 0$ such that for each $\delta > 0$, there exists $x \in A$ such that $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon_0$. Therefore, for each $n \in \mathbb{N}$, there exists $x_n \in A$ such that $0 < |x - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$. However, notice then that $(x_n) \subseteq A \setminus \{c\}$ is a sequence such that $x_n \to c$.

This implies $f(x_n) \to L$, which means there exists $N \in \mathbb{N}$ such that $|f(x_N) - L| < \varepsilon_0$, which is a contradiction.