Theorem (Ratio test) Consider the series $\sum a_n$ and let $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then

(i) if $L < 1$, the series converges absolutely

(ii) if $L > 1$, the series diverges

(iii) if $L = 1$, the test is inconclusive.

Proof (iii) Consider $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Both series yield $L = 1$ but one diverges and the other converges. Therefore, when $L = 1$, both behaviors are possible.

(i) Suppose $L < 1$ and let $\varepsilon > 0$ be given so that $L + \varepsilon < 1$. There exists N so that when $n > N$,

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - L \right| < \varepsilon.$$

Therefore, $\left| \frac{a_{n+1}}{a_n} \right| < L + \varepsilon$ when $n > N$.

Let $n > N$. Then

$$|a_n| = \left| \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+2}}{a_{N+1}} a_{N+1} \right|$$

$$< (L+\varepsilon)(L+\varepsilon) \cdots (L+\varepsilon) |a_{N+1}|$$

$$= (L+\varepsilon)^{n-(N+2)+1} |a_{N+1}|$$

$$= C(L+\varepsilon)^n$$

where $C = |a_{N+1}| (L+\varepsilon)^{-(N+1)}$

By comparison test, $\displaystyle\sum_{n=N+1}^{\infty} |a_n|$

converges since $\displaystyle\sum_{n=N+1}^{\infty} C(L+\varepsilon)^n$, since

it's a geometric series with ratio $L+\varepsilon < 1$.

Therefore $\sum a_n$ converges.

See More Series worksheet solutions

for other problems discussed.