§19 Uniform Continuity

(Warm up) Example Prove $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is continuous using an ε-δ proof.

Let $a \in \mathbb{R}$, $\varepsilon > 0$ and define $\delta = \min\{1, \frac{\varepsilon}{1+3|a|+3a^2}\}$. Suppose $x \in \mathbb{R}$ and $0 < |x-a| < \delta$. Then

$$\begin{aligned}
|f(x) - f(a)| &= |x^3 - a^3| \\
&= |x-a||x^2 + ax + a^2| \\
&< \delta |x^2 + ax + a^2| \\
&\leq \delta (|x|^2 + |a||x| + a^2)
\end{aligned}$$

Observe that

$$|x| = |x-a+a| \leq |x-a| + |a| < \delta + |a| \leq 1 + |a|$$

Therefore

$$\begin{aligned}
|f(x)-f(a)| &< \delta (|x|^2 + |a||x| + a^2) \\
&< \delta ((1+|a|)^2 + |a|(1+|a|) + a^2) \\
&= \delta (1 + 3|a| + 3a^2) \leq \varepsilon
\end{aligned}$$

Def Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be given.

Then f is <u>uniformly continuous</u> on D if

$\forall\, \varepsilon > 0 \ \exists\, \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$

for any $x, y \in D$ such that $|x - y| < \delta$.

Remarks ① if f is continuous on D, δ can depend on ε and a, but when f is <u>uniformly</u> continuous, δ only depends on ε.

② uniform continuity implies continuity but the converse is not true in general.

Example Prove $f(x) = x^3$ is uniformly continuous on $[-5, 5]$.

Let $\varepsilon > 0$ and define $\delta = \underline{\varepsilon / 75}$. Suppose $x, y \in [-5, 5]$ and $|x - y| < \delta$. Then

$$\begin{aligned}
|f(x) - f(y)| &= |x^3 - y^3| \\
&= |x - y|\,|x^2 + xy + y^2| \\
&< \delta\, |x^2 + xy + y^2| \\
&\leq \delta\, (|x|^2 + |x||y| + |y|^2) \\
&\leq \delta\, (25 + 25 + 25) \\
&= \varepsilon.
\end{aligned}$$

Problem 1. For each given domain D and function $f : D \to \mathbb{R}$, give an ϵ-δ proof to show that f is uniformly continuous on D.

a. $f(x) = x^2$, $D = [-4, 3]$.

b. $f(x) = 1/x$, $D = (2, 3)$.

c. $f(x) = 1/(x-3)$, $D = (6, \infty)$.

(a) Let $\varepsilon > 0$ and define $\delta = \varepsilon/8$.

Suppose $x, y \in [-4, 3]$ and $|x - y| < \delta$.

Then
$$
\begin{aligned}
|f(x) - f(y)| &= |x^2 - y^2| \\
&= |x-y||x+y| \\
&< \delta |x+y| \\
&\leq \delta (|x| + |y|) \\
&\leq 8\delta \\
&= \varepsilon
\end{aligned}
$$

(b) Let $\varepsilon > 0$ and define $\delta = 4\varepsilon$.

Suppose $x, y \in (2, 3)$ and $|x - y| < \delta$.

Then
$$
\begin{aligned}
|f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\
&= \left| \frac{y-x}{xy} \right| \\
&< \frac{\delta}{|x||y|} \\
&< \frac{\delta}{4} \\
&= \varepsilon.
\end{aligned}
$$

(i) Let $\varepsilon > 0$ and define $\delta = 9\varepsilon$.

Suppose $x, y \in (6, \infty)$ and $|x - y| < \delta$.

Then
$$|f(x) - f(y)| = \left| \frac{1}{x-3} - \frac{1}{y-3} \right|$$
$$= \left| \frac{y-3-(x-3)}{(x-3)(y-3)} \right|$$
$$= \left| \frac{y-x}{(x-3)(y-3)} \right|$$
$$< \frac{\delta}{|x-3||y-3|}$$
$$< \frac{\delta}{9}$$
$$= \varepsilon.$$