§ 4.4 Continuous functions on compact sets

A reminder of notation: given $f: A \to \mathbb{R}$ and $B \subseteq A$, the set $f(B) = \{f(x) : x \in B\}$ is the image of B under f.

In Calculus I you often think about finding max/min of a function $f: [a,b] \to \mathbb{R}$ that is continuous.

Why do these functions always have a max/min?

Theorem (Preservation of compact sets). Let $f: A \to \mathbb{R}$ be continuous on A and let $K \subseteq A$ be a compact set. Then $f(K)$ is compact.

Proof We must show that for each sequence $(y_n) \subseteq f(K)$, there exists a convergent subsequence (y_{n_k}) whose limit is in $f(K)$. Let $(y_n) \subseteq f(K)$. Then there exists a sequence $(x_n) \subseteq K$ such that $y_n = f(x_n)$ for all $n \in \mathbb{N}$.

Since K is compact, there exists a subsequence (x_{n_k}) with limit $x \in K$. Since f is continuous,

$$y_{n_k} = f(x_{n_k}) \to f(x) \in f(K).$$

Exercise 3.3.1 If K is compact, then $\sup K$ and $\inf K$ both exist and are elements of K.

Corollary If K is compact, $\max K$ and $\min K$ both exist.

Theorem (Extreme Value Theorem) If $f: K \to \mathbb{R}$ is continuous on a compact set K, then f attains a maximum value and a minimum value. More precisely, there exist $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

Proof By the preservation of compact sets, $f(K)$ is compact. By the previous corollary, $\max f(K)$ and $\min f(K)$ exist. Therefore there exist $y_0, y_1 \in f(K)$ such that $y_0 \leq y \leq y_1$ for all $y \in f(K)$. In other words, there exist $x_0, x_1 \in K$ such that

$$f(x_0) \leq f(x) \leq f(x_1)$$

for all $x \in K$.

Example Consider $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x+1$ and $g(x) = x^2$. Give ε-δ proofs showing they are continuous on $\mathbb{R}$ (ie continuous at c for every $c \in \mathbb{R}$). Pay attention to whether your δ depends on both ε and c or just ε.

(a) Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Define $\delta = \underline{\varepsilon/3}$.

Let $x \in \mathbb{R}$ with $|x-c| < \delta$. Then

$$\begin{aligned} |f(x) - f(c)| &= |3x+1-(3c+1)| \\ &= 3|x-c| \\ &< 3\delta \\ &= \varepsilon. \end{aligned}$$

Notice δ does <u>not</u> depend on c.

(b) Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Define $\delta = \underline{\min\{1, \frac{\varepsilon}{1+2|c|}\}}$.

Let $x \in \mathbb{R}$ with $|x-c| < \delta$. Then

$$
\begin{aligned}
|f(x) - f(c)| &= |x^2 - c^2| \\
&= |x-c||x+c| \\
&< \delta |x+c| \\
&\leq \delta (|x| + |c|) \\
&= \delta (|x-c+c| + |c|) \\
&\leq \delta (|x-c| + 2|c|) \\
&< \delta (\delta + 2|c|) \\
&\leq \delta (1 + 2|c|) \\
&= \varepsilon
\end{aligned}
$$

Notice our δ does depend on c.