Math 301 — Exam 2 review guide solutions

Solution 1.

a. *True*: if $\sum a_n$ converges, then (a_n) converges to 0. Since (a_n) converges it is Cauchy.

b. *False*: it is possible that x is an isolated point. For example, consider the set $A = \{x\}$, where $x \in \mathbb{R}$, and the sequence (a_n) where $a_n = x$ for all $n \in \mathbb{N}$. Then $a_n \to x$ but A has no limit points. *Modified statement that is true:* if $A \subseteq \mathbb{R}$ is a given set and there exists a sequence $(a_n) \subseteq A \setminus \{x\}$ such that $a_n \to x$, then x is a limit point of A.

c. *False*: consider the open sets given by $A_n = (0, 1+1/n)$ for all $n \in \mathbb{N}$. Notice that $\bigcap_{n=1}^{\infty} A_n = (0, 1]$ is not open. *Modified statement that is true:* if $A_1, A_2, \ldots$ are closed sets, then $\bigcap_{n=1}^{\infty} A_n$ is a closed set.

d. *True*: if x is an isolated point of A then $x \in A$ by definition. Therefore, denoting the set of limit points of A by L, we have that $x \in A \cup L = \overline{A}$.

e. *False*: consider the set $K = [-1, 1]$ and the sequence given by $x_n = (-1)^n$ for all $n \in \mathbb{N}$. Then K is compact but contains a sequence which does not converge. *Modified statement that is true:* if K is a compact set, then every sequence has a convergent subsequence.

f. *False*: let $\epsilon > 0$ and define $\delta = \epsilon$. Suppose that $x \in \mathbb{R}$ is such that $0 < |x| < \delta$. Then $|f(x)| = |x| < \delta = \epsilon$. Therefore $\lim_{x \to 0} f(x) = 0$. *Modified statement that is true:* let Let
$$f(x) = \begin{cases} 1/x & x \neq 0 \\ 1 & x = 0 \end{cases}. \text{ for all } x \neq 0. \text{ Then } \lim_{x \to 0} f(x) \text{ does not exist.}$$

Solution 2.

a. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} a_n$ converges absolutely, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$ we have that
$$\sum_{k=m+1}^{n} |a_k| < \epsilon.$$

Since (b_n) is a subsequence of (a_n) we have that there exists a strictly increasing sequence $(n_k) \subseteq \mathbb{N}$ such that $b_1 = a_{n_1}, b_2 = a_{n_2}, \ldots$. That is, $b_k = a_{n_k}$ for each $k \in \mathbb{N}$. Suppose that $i > j \geq N$. This implies that $n_i > n_j \geq N$. Therefore

$$\sum_{k=j+1}^{i} |b_k| = |b_{j+1}| + \cdots + |b_i|$$
$$= |a_{n_{j+1}}| + \cdots + |a_{n_i}|$$
$$\leq \sum_{k=n_{j+1}}^{n_i} |a_k|$$
$$< \epsilon.$$

By the Cauchy criterion for series, $\sum_{n=1}^{\infty} b_n$ converges absolutely.

b. Recall that a set A is open if for every $x \in A$ there exists $\epsilon > 0$ such that $V_\epsilon(x) \subseteq A$. Therefore it is not open if there exists $x \in A$ such that $V_\epsilon(x) \nsubseteq A$ for all $\epsilon > 0$. This is indeed the case here with $x = b$. Notice that if $\epsilon > 0$ then $V_\epsilon(b) \nsubseteq A$ because for every $y \in (b, b + \epsilon)$ we have that $y \in V_\epsilon(b)$ but $y \notin A$. Recall that a set A is closed if it contains all of its limit points.

Therefore it is not closed if there exists a limit point x of A which is not an element of A. That is indeed the case here with $x = a$. Notice that a is a limit point of $(a, b]$ since there exists a sequence $(x_n) \subseteq (a, b]$ such that $x_n \to a$. Indeed, $x_n = a + (b-a)/n$ for each $n \in \mathbb{N}$ is such a sequence.

c. Let $n \in \mathbb{N}$ and suppose that $K_1, \ldots, K_n \subseteq \mathbb{R}$ are compact sets. Since each set is compact, each set is closed. Therefore, $\bigcup_{i=1}^{n} K_i$ is closed. Since each set is compact, each set is bounded. That is, for each $i \in \{1, \ldots, n\}$ we have that there exists $M_i > 0$ such that $|x| \leq M_i$ for all $x \in K_i$. Let $M = \max\{M_1, \ldots, M_n\}$. Suppose that $x \in \bigcup_{i=1}^{n} K_i$. Then $x \in K_i$ for some $i \in \{1, \ldots, n\}$ which means that $|x| \leq M_i \leq M$. Therefore $\bigcup_{i=1}^{n} K_i$ is bounded. Since $\bigcup_{i=1}^{n} K_i$ is closed and bounded, it is compact.

d. Let $s_n = \sum_{k=1}^{n} 1/k^2$ for each $n \in \mathbb{N}$. Then $(s_n) \subseteq A$ is a sequence that converges to α. Moreover, it is clear that $s_n \neq \alpha$ for all $n \in \mathbb{N}$ since $s_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $\alpha \notin \mathbb{Q}$. Therefore α is a limit point of A but not an element A. This means A is not closed.

e. Let $\epsilon > 0$ and define $\delta = \min\{1, \epsilon/37\}$. Suppose that $x \in \mathbb{R}$ is such that $0 < |x - 3| < \delta$. Observe that

$$|x^3 - 27| = |x - 3||x^2 + 3x + 9|$$
$$< \delta|x^2 + 3x + 9|$$
$$\leq \delta(|x|^2 + 3|x| + 9).$$

Notice that since $\delta \leq 1$ and $|x - 3| < \delta$ it follows that $2 < x < 4$. Therefore $|x|^2 + 3|x| + 9 < 4^2 + 3(4) + 9 = 37$. Then we have that

$$|x^3 - 27| < 37\delta \leq \epsilon.$$

f. Note that since $\overline{\mathbb{Q}} = \mathbb{R}$ we have that there exists a sequence $(x_n) \subseteq \mathbb{Q}$ such that $x_n \to \sqrt{2}$. Moreover, if we let $y_n = \sqrt{2} + 1/n$ for each $n \in \mathbb{N}$ we note that $(y_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$, $y_n \neq \sqrt{2}$ for all $n \in \mathbb{N}$, and $y_n \to \sqrt{2}$. Finally, we note that $f(x_n) = 0$ and $f(y_n) = y_n$ for all $n \in \mathbb{N}$ which means that $\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$. Therefore, by the Divergence Criterion for Functional Limits, $\lim_{x \to \sqrt{2}} f(x)$ does not exist.