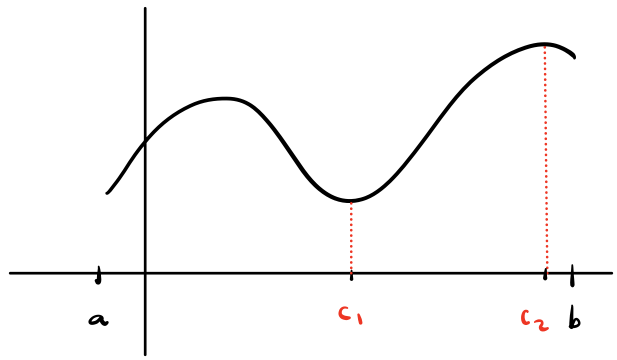
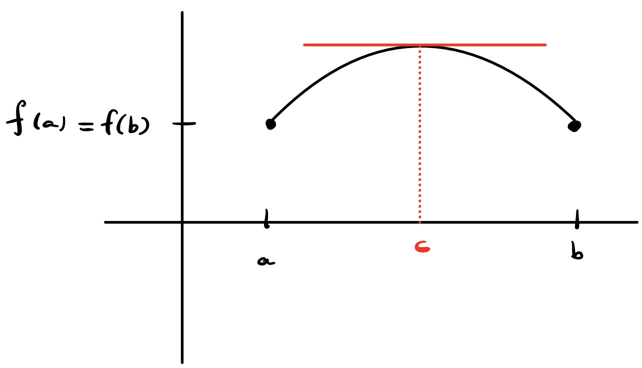
§ 29 Mean Value Theorem

Theorem 1 (Interior Extremum Theorem). *Let f be differentiable on the open interval (a, b). If f achieves a maximum value at some point $c \in (a, b)$, then $f'(c) = 0$. The same holds if $f(c)$ is a minimum value.*

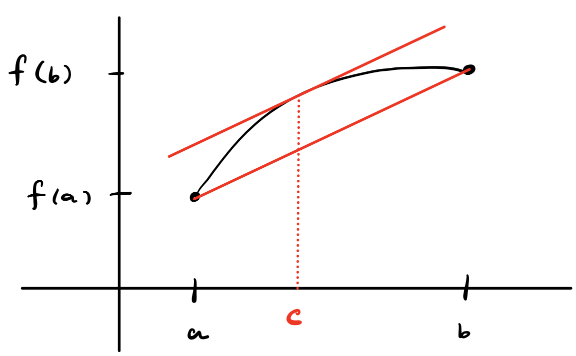


Theorem 2 (Rolle's Theorem). *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*



Theorem 3 (Mean Value Theorem). *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Problem 1. The following questions outline a sequential proof of the Interior Extremum Theorem. Assume that f is differentiable on the open interval (a, b) and assume that f achieves a maximum value at the point $c \in (a, b)$. That is, assume that $f(c) \geq f(x)$ for all $x \in (a, b)$. The case where $f(c)$ is a minimum value is similar and left for you to think about on your own.

a. Let $(x_n) \subseteq (a, c)$ and $(y_n) \subseteq (c, b)$ be sequences such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$.

1. Is the difference quotient
$$\frac{f(x_n) - f(c)}{x_n - c}$$
nonpositive or nonnegative for each $n \geq 1$? Try thinking about the sign of the numerator and denominator separately.

2. Is the difference quotient
$$\frac{f(y_n) - f(c)}{y_n - c}$$
nonpositive or nonnegative for each $n \geq 1$?

b. What do each of the previous parts tell you about the sign of $f'(c)$? Write a short explanation for why the result of the theorem now follows. Come back and write a full proof when you finish the rest of the worksheet.

(a) ① $f(x_n) \leq f(c) \quad \forall \; n \geq 1$

$\Rightarrow f(x_n) - f(c) \leq 0 \quad \forall \; n \geq 1$

and $x_n < c \quad \forall \; n \geq 1$

$\Rightarrow x_n - c < 0 \quad \forall \; n \geq 1$

Therefore $\dfrac{f(x_n) - f(c)}{x_n - c} \geq 0 \quad \forall \; n \geq 1$

② $f(y_n) \leq f(c) \quad \forall \; n \geq 1$

$\Rightarrow f(y_n) - f(c) \leq 0 \quad \forall \; n \geq 1$

and $y_n > c \quad \forall \; n \geq 1$

$\Rightarrow y_n - c > 0 \quad \forall \; n \geq 1$

Therefore $\dfrac{f(y_n) - f(c)}{y_n - c} \leq 0 \quad \forall \; n \geq 1$

(b) Since f is differentiable at c, $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists.

This implies $\exists \; L$ such that $\forall$ sequences (x_n) such that $\lim\limits_{n\to\infty} x_n = c$ and $x_n \neq c \; \forall \; n \geq 1$,

$\lim\limits_{n\to\infty} \dfrac{f(x_n) - f(c)}{x_n - c} = L$. By the discussion above, $L \leq 0$

and $L \geq 0$. Therefore $L = 0$. That is, $f'(c) = 0$.

Problem 2. The following questions outline a proof of Rolle's theorem. Assume that f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Recall that the Extreme Value Theorem tells us that there exist x_0 and y_0 in $[a, b]$ so that

$$f(y_0) \leq f(x) \leq f(x_0)$$

for all $x \in [a, b]$.

a. Suppose at least one of x_0 or y_0 is in (a, b). Why does there exist $c \in (a, b)$ such that $f'(c) = 0$.

b. Suppose both x_0 and y_0 occur at the endpoints of $[a, b]$. Why does there exist $c \in (a, b)$ such that $f'(c) = 0$.

(a) If $x_0 \in (a,b)$, by the Interior Extremum Theorem, $f'(x_0) = 0$. If $y_0 \in (a,b)$, $f'(y_0) = 0$ similarly. In either case, $\exists$ $c \in (a,b)$ such that $f'(c) = 0$.

(b) Suppose $x_0 = a$ and $y_0 = b$. Then $\forall$ $x \in [a,b]$

$$f(b) \leq f(x) \leq f(a) = f(b)$$

Then $f(x)$ is constant $\forall$ $x \in [a,b]$ and $f'(x) = 0$ $\forall$ $x \in [a,b]$. The same holds if $x_0 = b$, $y_0 = a$ or $x_0 = a$, $y_0 = a$ or $x_0 = b$, $y_0 = b$.

Problem 3. We now attempt to prove the Mean Value Theorem.

a. Let $L(x)$ be the secant line that connects the points $(a, f(a))$ and $(b, f(b))$ on the graph of f. Give the value of $L'(x)$ for all $x \in (a, b)$.

b. Let $g(x) = f(x) - L(x)$. Explain why g satisfies the hypotheses of Rolle's Theorem.

c. Use the previous parts and Rolle's Theorem to prove the Mean Value Theorem.

(a) $L'(x) = \dfrac{f(b) - f(a)}{b - a}$

(b) Since $L(a) = f(a)$ and $L(b) = f(b)$, $g(a) = g(b) = 0$. Since f and L are continuous on $[a,b]$ and differentiable on (a,b), g is too.

(c) By Rolle's Theorem, $\exists\ c \in (a,b)$ such that $g'(c) = 0$. Therefore, $f'(c) - L'(c) = 0$, which implies $f'(c) = L'(c) = \dfrac{f(b) - f(a)}{b - c}$.