

§29 Mean Value Theorem, FTOC

The MVT serves as the foundation for a number of important ideas from calculus as we'll now discuss.

Corollary Let $f: (a,b) \rightarrow \mathbb{R}$ be differentiable. If $f'(x)=0 \quad \forall x \in (a,b)$, then f is constant on (a,b) .

Proof Suppose that f is not constant on (a,b) .

Then $\exists x, y \in (a,b)$ with $x \neq y$ and $f(x) \neq f(y)$.

Suppose without loss of generality that $x < y$.

Then f is continuous on $[x,y]$ and differentiable

on (x,y) and by the MVT, $\exists c \in (x,y)$

such that $\frac{f(x)-f(y)}{x-y} = f'(c)$.

Since $f(x) \neq f(y)$, we've shown $\exists c \in (a,b)$ such that

$$f'(c) \neq 0.$$

Corollary Let $f, g: (a,b) \rightarrow \mathbb{R}$ be differentiable. If

$f' = g'$, then $\exists C \in \mathbb{R}$ such that $f(x) = g(x) + C$

for all $x \in (a,b)$.

Proof Let $h: (a,b) \rightarrow \mathbb{R}$ be given by $h(x) = f(x) - g(x)$.

Then h is differentiable and $h'(x) = 0 \quad \forall x \in (a,b)$

By the previous corollary, h is constant on (a,b) .

That is, $\exists C \in \mathbb{R}$ such that $h(x) = C \quad \forall x \in (a,b)$.

The result follows.

Def Let I be an interval and $f: I \rightarrow \mathbb{R}$ a given function. Then f is strictly increasing on I if $x_1, x_2 \in I$ and $x_1 < x_2$ implies $f(x_1) < f(x_2)$. We say f is increasing on I if $x_1, x_2 \in I$ and $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$. Similar definitions hold for decreasing and strictly decreasing on I .

Theorem Let $f: (a,b) \rightarrow \mathbb{R}$ be differentiable. Then

- ① f is strictly increasing if $f'(x) > 0 \quad \forall x \in (a,b)$.
- ② f is increasing if $f'(x) \geq 0 \quad \forall x \in (a,b)$.
- ③ f is strictly decreasing if $f'(x) < 0 \quad \forall x \in (a,b)$.
- ④ f is decreasing if $f'(x) \leq 0 \quad \forall x \in (a,b)$.

Proof of ①

Let $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$. By the MVT, $\exists c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c),$$

which implies $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) > 0$ and $x_2 - x_1 > 0$, $f(x_2) - f(x_1) > 0$ and so $f(x_2) > f(x_1)$.

Problem 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

Let $a \in \mathbb{R}$. We claim $f'(a) = 0$. Indeed, let

(x_n) be an arbitrary sequence such that $x_n \neq a$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} x_n = a$. Then

$$|f(x_n) - f(a)| \leq (x_n - a)^2 \text{ for all } n \geq 1, \text{ which implies}$$

$$0 \leq \left| \frac{f(x_n) - f(a)}{x_n - a} \right| \leq |x_n - a|$$

for all $n \geq 1$. By the Squeeze Theorem, $\lim_{n \rightarrow \infty} \left| \frac{f(x_n) - f(a)}{x_n - a} \right| = 0$

which implies $f'(a) = 0$. By the first corollary to the MVT, f is a constant.

Problem 2. Let $f : I \rightarrow \mathbb{R}$ be given. Suppose that f is twice differentiable on I and $f''(x) = 0$ for all $x \in I$. Prove that $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Since $f''(x) = 0 \quad \forall x \in I$, $\exists a \in \mathbb{R}$ such that

$f'(x) = a \quad \forall x \in I$ by the first corollary of the MVT.

Let $g(x) = ax \quad \forall x \in I$. Then $g' = f'$ and by the

second corollary of the MVT, $\exists b \in \mathbb{R}$ so that

$f(x) = g(x) + b \quad \forall x \in I$. Therefore $f(x) = ax + b$.

Problem 3. Prove that $\sin x \leq x$ for all $x \geq 0$.

Let $f(x) = \sin x - x$. Then $f'(x) = \cos x - 1$. Since

$\cos x \leq 1 \quad \forall x \geq 0$, $f'(x) \leq 0 \quad \forall x \geq 0$. Therefore f

is decreasing on $[0, \infty)$ and $f(0) = 0$, which implies

$f(x) \leq 0$ for all $x \geq 0$. Thus $\sin x \leq x$ for all $x \geq 0$.

Theorem (Intermediate Value Theorem for Derivatives)

Let $f: (a,b) \rightarrow \mathbb{R}$ be differentiable. If $x_1, x_2 \in (a,b)$ with $x_1 < x_2$ and c is between $f'(x_1)$ and $f'(x_2)$ then $\exists x_0 \in (x_1, x_2)$ with $f'(x_0) = c$.

Proof Suppose without loss of generality that $f'(x_1) < f'(x_2)$ and $c \in (f'(x_1), f'(x_2))$. Define $g(x) = f(x) - cx$.

We aim to show $\exists x_0 \in (x_1, x_2)$ such that $g'(x_0) = 0$, which will imply the desired result since $g'(x) = f'(x) - c$. Since g is differentiable on (a,b) it's continuous on $[x_1, x_2]$ and achieves its minimum at some $x_0 \in [x_1, x_2]$ by the Extreme Value Theorem. By the Interior Extremum Theorem it suffices to show

$x_0 \in (x_1, x_2)$ since this implies $g'(x_0) = 0$. We first show $x_0 \neq x_1$. Notice $g'(x_1) = f'(x_1) - c < 0$.

Therefore $\lim_{y \rightarrow x_1} \frac{g(y) - g(x_1)}{y - x_1} < 0$. In particular

if $(y_n) \subseteq (x_1, x_2)$ is a sequence converging to x_1 , $\lim_{n \rightarrow \infty} \frac{g(y_n) - g(x_1)}{y_n - x_1} < 0$ implies $\exists y \in (x_1, x_2)$ so that $g(y) - g(x_1) < 0$, which implies x_1 is not a minimum. Similarly, notice $g'(x_2) = f'(x_2) - c > 0$.

Therefore $\lim_{y \rightarrow x_2} \frac{g(y) - g(x_2)}{y - x_2} > 0$. In particular

if $(y_n) \subseteq (x_1, x_2)$ is a sequence converging to x_2 ,

$\lim_{n \rightarrow \infty} \frac{g(y_n) - g(x_2)}{y_n - x_2} > 0$. This implies $\exists y' \in (x_1, x_2)$

such that $g(y') - g(x_2) < 0$. This implies x_2 is not a minimum either and so $x_2 \neq x_0$.