§ 34 FToC

Theorem (Fundamental Theorem of Calculus, part II)

Let $f: [a,b] \to \mathbb{R}$ be continuous. Define $g: [a,b] \to \mathbb{R}$ by $g(x) = \int_a^x f(t)\,dt$. Then g is differentiable on (a,b) and $g'(x) = f(x)$.

Proof Let $x_0 \in (a,b)$ and $\varepsilon > 0$. We aim to show $\exists\ \delta > 0$ such that

$$\left| \frac{g(x) - g(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon$$

when $x \in (a,b)$ and $0 < |x - x_0| < \delta$. Since f is uniformly continuous on $[a,b]$, $\exists\ \delta > 0$ such that $\left| f(x) - f(y) \right| < \varepsilon$ when $|x-y| < \delta$.

Suppose $x \in (a,b)$ and $0 < |x - x_0| < \delta$. Observe that

$$\left| \frac{g(x) - g(x_0)}{x - x_0} - f(x_0) \right|$$

$$= \left| \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} - f(x_0) \right|$$

$$= \left| \frac{\int_{x_0}^{x} f(t)\,dt}{x - x_0} - f(x_0) \right|$$

$$= \left| \frac{\int_{x_0}^{x} f(t)\,dt}{x - x_0} - \frac{f(x_0)\,(x - x_0)}{x - x_0} \right|$$

$$= \left| \frac{\int_{x_0}^{x} f(t)\,dt}{x - x_0} - \frac{\int_{x_0}^{x} f(x_0)\,dt}{x - x_0} \right|$$

$$= \left| \frac{\int_{x_0}^{x} \big(f(t) - f(x_0)\big)\,dt}{x - x_0} \right|$$

$$\leq \frac{1}{|x - x_0|} \int_{x_0}^{x} |f(t) - f(x_0)|\,dt \quad \text{by triangle inequality}$$

$$< \frac{1}{|x - x_0|} \int_{x_0}^{x} \varepsilon\,dt$$

$$= \varepsilon.$$

Def Let $f : I \to \mathbb{R}$ be a given function.

A function $F : I \to \mathbb{R}$ is called an antiderivative of f on I if $F'(x) = f(x)$ $\forall\, x \in I$.

Theorem (FTOC, part I). Let $f : I \to \mathbb{R}$ be continuous and F any antiderivative of f.

Then for any $a, b \in I$, $\int_a^b f(x)\,dx = F(b) - F(a)$.

Problem 1. Let $f : I \to \mathbb{R}$ be a given function. Prove that if $F, G : I \to \mathbb{R}$ are antiderivatives of f then there exists $C \in \mathbb{R}$ such that $F(x) = G(x) + C$ for all $x \in I$.

Since $F' = f$ and $G' = f$, we have $F' = G'$ and by the second corollary of the MVT, $\exists\ C \in \mathbb{R}$ so that $F(x) = G(x) + C$ for all $x \in I$.

Problem 2. Let $f : I \to \mathbb{R}$ be a continuous function and let $F : I \to \mathbb{R}$ be an antiderivative of f. Prove that for any $a, b \in I$

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

By the FTOC, part II, $g(x) = \int_a^x f(t)\,dt$ is an antiderivative of f. By the previous problem, $\exists\ C \in \mathbb{R}$ such that $F(x) = g(x) + C$. Therefore

$$\begin{aligned} F(b) - F(a) &= (g(b) + C) - (g(a) + C) \\ &= g(b) - g(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

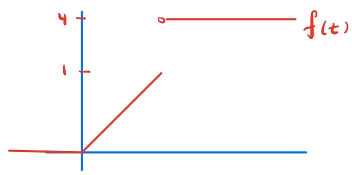
$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 4 & t > 1. \end{cases}$$

a. Find a formula for $F(x) = \int_0^x f(t)\,dt$.

b. Make a sketch of the graph of F and make a conjecture of where F is continuous.

c. Make a conjecture of where F is differentiable and find $F'(x)$ for all x where you believe F is differentiable.

d. In casual conversation, an analyst might say *integration has a smoothing effect* or *integration increases regularity.* What do they mean by this?



(a) $F(x) = 0$ when $x < 0$

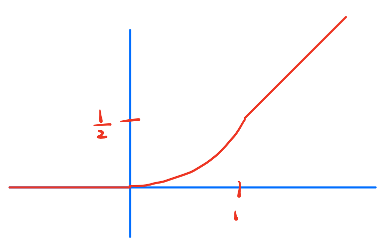
if $0 \le x \le 1$, $F(x) = \int_0^x t\,dt$

$= \frac{1}{2}x^2$

if $x > 1$, $F(x) = \int_0^1 t\,dt + \int_1^x 4\,dt$

$= \frac{1}{2} + 4(x-1)$

(b)



F is continuous on $\mathbb{R}$

(c) $F(x)$ is differentiable everywhere except $x = 1$.

$$F'(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 4 & x > 1 \end{cases}$$

$= f(x)$ when $x \ne 1$.