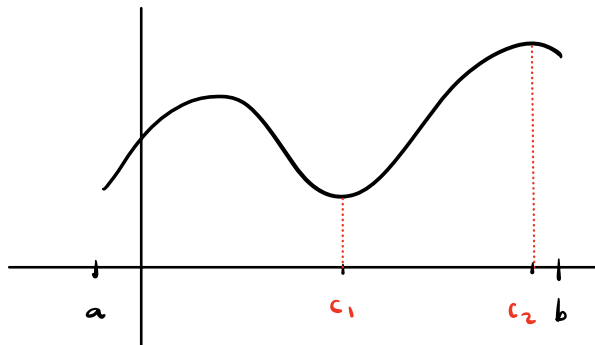
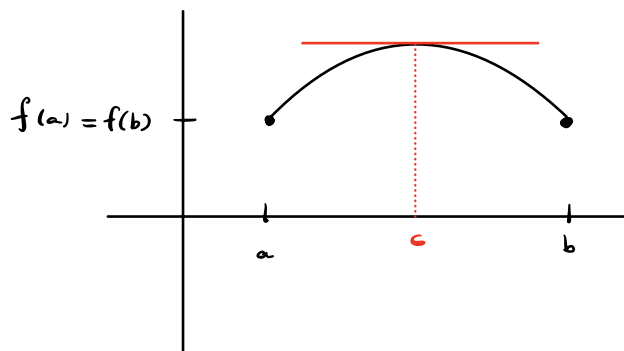


§5.3 Mean Value Theorem

Theorem 1 (Interior Extremum Theorem). *Let f be differentiable on the open interval (a, b) . If f achieves a maximum value at some point $c \in (a, b)$, then $f'(c) = 0$. The same holds if $f(c)$ is a minimum value.*

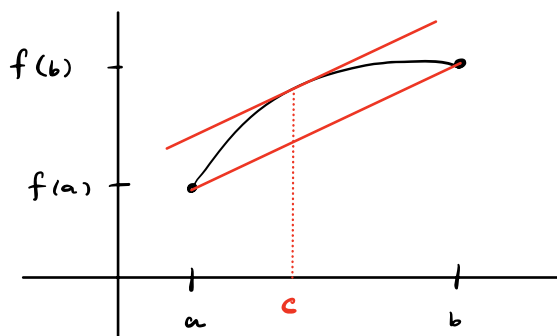


Theorem 2 (Rolle's Theorem). *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = f(b)$ then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*



Theorem 3 (Mean Value Theorem). *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b) . Then there exists a point $c \in (a, b)$ so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Problem 1. The following questions outline a sequential proof of the Interior Extremum Theorem. Assume that f is differentiable on the open interval (a, b) and assume that f achieves a maximum value at the point $c \in (a, b)$. That is, assume that $f(c) \geq f(x)$ for all $x \in (a, b)$. The case where $f(c)$ is a minimum value is similar and left for you to think about on your own.

- a. Let $(x_n) \subseteq (a, c)$ and $(y_n) \subseteq (c, b)$ be sequences such that $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = c$.

1. Is the difference quotient

$$\frac{f(x_n) - f(c)}{x_n - c}$$

nonpositive or nonnegative for each $n \geq 1$? Try thinking about the sign of the numerator and denominator separately.

2. Is the difference quotient

$$\frac{f(y_n) - f(c)}{y_n - c}$$

nonpositive or nonnegative for each $n \geq 1$?

- b. What do each of the previous parts tell you about the sign of $f'(c)$? Write a short explanation for why the result of the theorem now follows. Come back and write a full proof when you finish the rest of the worksheet.

$$\begin{aligned} \textcircled{a} \quad \textcircled{1} \quad & f(x_n) \leq f(c) \quad \forall \quad n \geq 1 \\ \Rightarrow & f(x_n) - f(c) \leq 0 \quad \forall \quad n \geq 1 \\ \text{and} \quad & x_n < c \quad \forall \quad n \geq 1 \\ \Rightarrow & x_n - c < 0 \quad \forall \quad n \geq 1 \end{aligned}$$

$$\text{Therefore} \quad \frac{f(x_n) - f(c)}{x_n - c} \geq 0 \quad \forall \quad n \geq 1$$

$$\begin{aligned} \textcircled{a} \quad \textcircled{2} \quad & f(y_n) \leq f(c) \quad \forall \quad n \geq 1 \\ \Rightarrow & f(y_n) - f(c) \leq 0 \quad \forall \quad n \geq 1 \\ \text{and} \quad & y_n > c \quad \forall \quad n \geq 1 \\ \Rightarrow & y_n - c > 0 \quad \forall \quad n \geq 1 \end{aligned}$$

$$\text{Therefore} \quad \frac{f(y_n) - f(c)}{y_n - c} \leq 0 \quad \forall \quad n \geq 1$$

$$\textcircled{b} \quad \text{Since } f \text{ is differentiable at } c, \quad \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \text{ exists.}$$

This implies $\exists L$ such that \forall sequences (x_n)

such that $\lim_{n \rightarrow \infty} x_n = c$ and $x_n \neq c \quad \forall \quad n \geq 1$,

$$\lim_{n \rightarrow \infty} \frac{f(x_n) - f(c)}{x_n - c} = L. \quad \text{By the discussion above, } L \leq 0$$

and $L \geq 0$. Therefore $L = 0$. That is, $f'(c) = 0$.

Problem 2. The following questions outline a proof of Rolle's theorem. Assume that f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$. Recall that the Extreme Value Theorem tells us that there exist x_0 and y_0 in $[a, b]$ so that

$$f(y_0) \leq f(x) \leq f(x_0)$$

for all $x \in [a, b]$.

- Suppose at least one of x_0 or y_0 is in (a, b) . Why does there exist $c \in (a, b)$ such that $f'(c) = 0$.
- Suppose both x_0 and y_0 occur at the endpoints of $[a, b]$. Why does there exist $c \in (a, b)$ such that $f'(c) = 0$.

Ⓐ If $x_0 \in (a, b)$, by the Interior Extremum Theorem,
 $f'(x_0) = 0$. If $y_0 \in (a, b)$, $f'(y_0) = 0$ similarly.
 In either case, $\exists c \in (a, b)$ such that $f'(c) = 0$.

Ⓑ Suppose $x_0 = a$ and $y_0 = b$. Then $\forall x \in [a, b]$

$$f(b) \leq f(x) \leq f(a) = f(b)$$

Then $f(x)$ is constant $\forall x \in [a, b]$ and $f'(x) = 0$

$\forall x \in [a, b]$. The same holds if $x_0 = b, y_0 = a$

or $x_0 = a, y_0 = a$ or $x_0 = b, y_0 = b$.

Problem 3. We now attempt to prove the Mean Value Theorem.

- Let $L(x)$ be the secant line that connects the points $(a, f(a))$ and $(b, f(b))$ on the graph of f . Give the value of $L'(x)$ for all $x \in (a, b)$.
- Let $g(x) = f(x) - L(x)$. Explain why g satisfies the hypotheses of Rolle's Theorem.
- Use the previous parts and Rolle's Theorem to prove the Mean Value Theorem.

$$\textcircled{a} \quad L'(x) = \frac{f(b) - f(a)}{b - a}$$

\textcircled{b} Since $L(a) = f(a)$ and $L(b) = f(b)$,
 $g(a) = g(b) = 0$. Since f and L are
continuous on $[a, b]$ and differentiable on (a, b) ,
 g is too.

\textcircled{c} By Rolle's Theorem, $\exists c \in (a, b)$ such that
 $g'(c) = 0$. Therefore, $f'(c) - L'(c) = 0$, which
implies $f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a}$.