§5.4 MVT

The MVT serves as the foundation for a number of important ideas from calculus as we'll now discuss.

Corollary Let $f:(a,b)\to\mathbb{R}$ be differentiable. If $f'(x)=0$ $\forall x\in(a,b)$, then f is constant on (a,b).

Proof Let $c,d\in(a,b)$ with $c<d$. Then f is continuous on $[c,d]$ and differentiable on (c,d). By the Mean Value Theorem, there exists $\alpha\in(c,d)$ such that $f'(\alpha)=\dfrac{f(c)-f(d)}{c-d}$. Since $f'(\alpha)=0$ we have $f(c)=f(d)$. Since c,d were arbitrary, f is constant on (a,b).

Corollary Let $f,g:(a,b)\to\mathbb{R}$ be differentiable. If $f'=g'$, then $\exists$ $C\in\mathbb{R}$ such that $f(x)=g(x)+C$ for all $x\in(a,b)$.

Proof Let $h:(a,b)\to\mathbb{R}$ be given by $h(x)=f(x)-g(x)$. Then h is differentiable and $h'(x)=0$ $\forall x\in(a,b)$. By the previous corollary, h is constant on (a,b). That is, $\exists$ $C\in\mathbb{R}$ such that $h(x)=C$ $\forall x\in(a,b)$. The result follows.

Example Let $f: \mathbb{R} \to \mathbb{R}$ be a function where $|f(x) - f(y)| \leq |x-y|^2$ for all $x, y \in \mathbb{R}$. Prove f is a constant.

We'll prove $f'(c) = 0$ for all $c \in \mathbb{R}$. The result will follow from the first corollary above. Let $c \in \mathbb{R}$. Let $\varepsilon > 0$. Define $\delta = \varepsilon$. Suppose $x \in \mathbb{R}$ and $0 < |x - c| < \delta$. Then

$$\left| \frac{f(x) - f(c)}{x - c} \right| \leq \frac{|x-c|^2}{|x-c|} = |x - c| < \delta = \varepsilon.$$

Therefore $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$, which means $f'(c) = 0$.

Example Let $f: \mathbb{R} \to \mathbb{R}$ be twice differentiable. Suppose $f''(x) = 0$ for all $x \in \mathbb{R}$. Prove that $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Since $f''(x) = 0$ $\forall x \in \mathbb{R}$, $\exists$ $a \in \mathbb{R}$ such that $f'(x) = a$ $\forall x \in \mathbb{R}$ by the first corollary of the MVT. Let $g(x) = ax$ $\forall x \in \mathbb{R}$. Then $g' = f'$ and by the second corollary of the MVT, $\exists$ $b \in \mathbb{R}$ so that $f(x) = g(x) + b$ $\forall x \in \mathbb{R}$. Therefore $f(x) = ax + b$.

Example Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$ using the Mean Value Theorem and the fact that $\frac{d}{dx}(\sin x) = \cos x$.

Let $x, y \in \mathbb{R}$. If $x = y$, the result is clearly true. Suppose $x < y$. By the MVT, there exists $c \in (x, y)$ such that

$$\sin x - \sin y = \cos c \, (x - y).$$

Since $|\cos c| \leq 1$, the result follows.

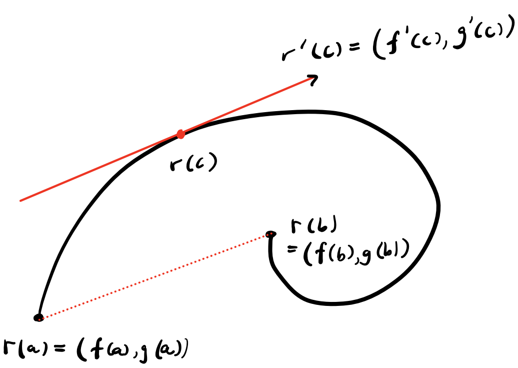
Theorem (Generalized Mean Value Theorem) Let f, g be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c).$$

Note if $g'(x) \neq 0$ for all $x \in (a,b)$, then there exists $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Intuition Let $r : [a,b] \to \mathbb{R}^2$ is the parametrized curve given by $r(t) = (f(t), g(t))$.



The theorem says there exists a point on the curve $r(t)$ whose tangent slope is the slope of the line connecting $r(a)$ and $r(b)$

Proof Let $h(x) = (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x)$.

Then h is continuous on $[a,b]$ and differentiable on (a,b).

Moreover, $h(a) = h(b)$. By Rolle's theorem, there exists

$c \in (a,b)$ such that $h'(c) = 0$. The result follows.

Theorem (L'Hôpital's Rule $\frac{0}{0}$ case) Let f, g be continuous on $[a,b]$ and differentiable on (a,b). Suppose further that $g'(x) \neq 0$ for all $x \neq c$ and $f(c) = g(c) = 0$. Then $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$ implies $\lim_{x \to c} \frac{f(x)}{g(x)} = L$.

Proof To show $\lim_{x \to c} \frac{f(x)}{g(x)} = L$ we can show that for every sequence $x_n \subseteq [a,b] \setminus \{c\}$ such that $x_n \to c$ we have that $\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = L$. Let (x_n) be such a sequence. Notice that for each $n \in \mathbb{N}$ there exists $c_n \in (x_n, c)$ or $c_n \in (c, x_n)$ such that

$$\frac{f'(c_n)}{g'(c_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f(x_n)}{g(x_n)}$$

by the Generalized Mean Value Theorem.

Notice that since $x_n \to c$, it follows that $c_n \to c$.

Therefore

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \lim_{n \to \infty} \frac{f'(c_n)}{g'(c_n)} = L$$