

## § 25 Properties of Uniform Convergence

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Theorem If  $f_n \rightarrow f$  uniformly on  $D$  and  $f_n$  is continuous on  $D \ \forall n \geq 1$ , then  $f$  is continuous on  $D$ .

Proof Let  $a \in D$  and  $\varepsilon > 0$ . We must show  $\exists \delta > 0$  such that  $|f(x) - f(a)| < \varepsilon$  when  $x \in D$  and  $|x - a| < \delta$ . Since  $f_n \rightarrow f$  uniformly,  $\exists N > 0$  such that  $|f_n(x) - f(x)| < \varepsilon/3$  for all  $n > N$  and  $x \in D$ . Since  $f_{N+1}$  is continuous at  $a$ ,  $\exists \delta > 0$  such that  $|f_{N+1}(x) - f_{N+1}(a)| < \varepsilon/3$  when  $x \in D$  and  $|x - a| < \delta$ . Suppose  $x \in D$  and  $|x - a| < \delta$ . Then

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f_{N+1}(x) + f_{N+1}(x) - f_{N+1}(a) + f_{N+1}(a) - f(a)| \\ &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(a)| + |f_{N+1}(a) - f(a)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

This is the "famous"  $\varepsilon/3$ -argument.

Theorem If  $f_n \rightarrow f$  uniformly on  $[a, b]$  and  $f_n, f$  integrable on  $[a, b] \forall n \geq 1$ , then

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$$

$f$  integrable means  $\int_a^b f(x) dx$  is finite

Proof Let  $\varepsilon > 0$ . Since  $f_n \rightarrow f$  uniformly on  $[a, b]$

$\exists N > 0$  so that  $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \forall n > N, \forall x \in [a, b]$ .

Suppose  $n > N$ . Then

$$\begin{aligned} \left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| &= \left| \int_a^b (f_n(x) - f(x)) dx \right| \\ &\leq \int_a^b |f_n(x) - f(x)| dx \\ &< \int_a^b \frac{\varepsilon}{b-a} dx = \varepsilon. \end{aligned}$$

**Problem 1.** We have proved a theorem that states:

If  $f_n \rightarrow f$  uniformly on  $D$  and  $f_n$  is continuous on  $D$  for all  $n \geq 1$ , then  $f$  is continuous on  $D$ .

State the contrapositive of this theorem.

If  $f$  is not continuous on  $D$  then either  $f_n$  is not continuous on  $D$  for some  $n \geq 1$  or  $(f_n)$  does not converge uniformly to  $f$  on  $D$ .

**Problem 2.** Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be given by  $f_n(x) = x^n$  for all  $n \geq 1$ . Let  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  for all  $x \in [0, 1]$ . Prove that  $(f_n)$  does not converge uniformly to  $f$  on  $[0, 1]$ .

Note  $f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$  is not continuous on  $[0, 1]$ .

Since  $f_n$  is continuous on  $[0, 1] \forall n \geq 1$ , we can

conclude  $f_n \not\rightarrow f$  uniformly on  $D$  by Problem 1.

**Problem 3.** We have proved a theorem that states:

If  $f_n \rightarrow f$  uniformly on  $[a, b]$  and  $f_n$  and  $f$  are integrable on  $[a, b]$  for all  $n \geq 1$ ,  
then  $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$ .

State the contrapositive of this theorem.

If  $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx \neq \int_a^b f(x) dx$ , then either  
 $f$  or  $f_n$  is not integrable for some  $n \geq 1$  or  
 $(f_n)$  does not converge uniformly to  $f$  on  $D$ .

**Problem 4.** Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be given by

$$f_n(x) = \begin{cases} nx^n & x \neq 1 \\ 0 & x = 1 \end{cases}$$

for all  $n \geq 1$ .

- Find  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  for all  $x \in [0, 1]$ .
- Let  $n \geq 1$ . Compute  $\int_0^1 f_n(x) dx$  in terms of  $n$ .
- Prove that  $(f_n)$  does not converge uniformly to  $f$  on  $[0, 1]$ .

(a)  $f(x) = 0 \quad \forall x \in [0, 1]$ .

(b)  $\int_0^1 f_n(x) dx = \int_0^1 nx^n dx = \frac{n}{n+1}$

(c) Note that  $\int_0^1 f(x) dx = 0 \neq 1 = \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$ .

Since  $f_n$  and  $f$  are integrable on  $[0, 1]$  for all  
 $n \geq 1$ , it must be that  $f_n \not\rightarrow f$  uniformly on  
 $[0, 1]$  by Problem 3.