Immediate corollaries of MVT

Corollary 1 If $f:(a,b) \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

Proof Suppose $\exists$ $x_1 \neq x_2 \in (a,b)$ where $f(x_1) \neq f(x_2)$.

Then $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad \text{for some } c \in (x_1, x_2)$$

by MVT. Since $f(x_1) \neq f(x_2)$, $f'(c) \neq 0$, which is a contradiction.

Corollary 2 If $f, g : (a,b) \to \mathbb{R}$ are differentiable and $f' = g'$, then $\exists$ $C \in \mathbb{R}$ such that $f(x) = g(x) + C$ for all $x \in (a,b)$.

Proof Let $h(x) = f(x) - g(x)$

Then $h'(x) = f'(x) - g'(x) = 0$

$\Rightarrow$ h is a constant by Corollary 1

$\Rightarrow$ $\exists$ $C \in \mathbb{R}$, $C = f(x) - g(x)$ for all x