§6.2 Uniform Convergence

$$\lim_{x\to c} f(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x) \overset{?}{=} \lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{n\to\infty} f_n(c) = f(c)$$

Theorem (Continuous Limit Theorem) Let (f_n) be a sequence of functions that are all continuous at $c \in A$. Suppose $f_n \to f$ uniformly on A. Then f is continuous at c.

Proof. Let $\varepsilon > 0$. We must show there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all $x \in A$ such that $|x - c| < \delta$.

Since $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < \varepsilon/3$ for all $x \in A$.

Since f_N is continuous at c, there exists $\delta > 0$ such that $|f_N(x) - f(c)| < \varepsilon/3$ for all $x \in A$ such that $|x - c| < \delta$. Therefore

$$\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\
&\le |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

<u>Example</u> Let $f_n(x) = x^n$ for all $x \in [0,1]$ and $n \in \mathbb{N}$.

Let $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$. Prove that (f_n) does not converge uniformly to f.

Since each f_n is continuous at 1 but f is not continuous at 1, (f_n) does not converge uniformly to f by the Continuous Limit Theorem.

<u>Theorem</u> (Bounded Limit Theorem) Let (f_n) be a sequence of functions that are all bounded on A. Suppose $f_n \to f$ uniformly on A. Then f is bounded on A.

<u>Proof</u> Since (f_n) are all bounded on A, for each $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|f_n(x)| \leq M_n$ for all $x \in A$. Since $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ so that $|f_N(x) - f(x)| < 1$ for all $x \in A$. To prove f is bounded on A, we must show that there exists $M > 0$ so that $|f(x)| \leq M$ for all $x \in A$. Let $x \in A$ and define $M = 1 + M_N$. Then

$$
\begin{aligned}
|f(x)| &= |f(x) - f_N(x) + f_N(x)| \\
&\leq |f_N(x) - f(x)| + |f_N(x)| \\
&< 1 + M_N \\
&= M.
\end{aligned}
$$

Theorem (Cauchy criterion for uniform converges)

Let (f_n) be a sequence of functions. Then (f_n) converges uniformly (to some f) on A if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_m(x) - f_n(x)| < \varepsilon \quad \text{for all } n, m \geq N \text{ and all } x \in A.$$

Proof. Exercise.