

§25 More on uniform convergence

Def Let (f_n) be a sequence of functions with domain D . Then (f_n) is uniformly Cauchy on D if $\forall \epsilon > 0 \exists N > 0$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m > N, x \in D$.

Theorem Let (f_n) be a sequence of functions with domain D . Then (f_n) converges uniformly on D if and only if it's uniformly Cauchy on D .

Proof (\Rightarrow) Assume $f_n \rightarrow f$ uniformly on D . Let $\epsilon > 0$. Then $\exists N > 0$ such that $|f_n(x) - f(x)| < \epsilon/2$ $\forall n > N, \forall x \in D$. Let $n, m > N$ and $x \in D$. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

\Leftrightarrow Assume (f_n) is uniformly Cauchy on D . Let $\epsilon > 0$. Then $\exists N > 0$ such that $\forall x \in D, \forall n, m > N$ $|f_n(x) - f_m(x)| < \epsilon/2$. This implies that $\forall x \in D$, $(f_n(x)) \subseteq \mathbb{R}$ is a Cauchy sequence and hence converges. So $\forall x \in D$, define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. We claim $f_n \rightarrow f$ uniformly on D . Let $n > N$ and $x \in D$. Then $\forall m > N$,

$$|f_n(x) - f_m(x)| < \epsilon/2,$$

which implies by the order limit theorem,

$$\lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \epsilon/2.$$

Therefore

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) - \lim_{m \rightarrow \infty} f_m(x)| \\ &= \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \\ &\leq \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

Def Let (g_n) be a sequence of functions on D and let $s_n(x) = \sum_{k=1}^n g_k(x) \quad \forall x \in D, \forall n \geq 1$. Then $\sum_{n=1}^{\infty} g_n$ converges uniformly if (s_n) converges uniformly.

Theorem (Weierstrass M-test) Let (g_n) be a sequence of functions with domain D and let $(M_n) \subseteq \mathbb{R}$ be a sequence of non-negative real numbers such that $\forall n \geq 1, |g_n(x)| \leq M_n \quad \forall x \in D$ and $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} g_n$ converges uniformly on D .

Proof Let $s_n = \sum_{k=1}^n g_k$ for each $n \geq 1$.

To show $\sum_{n=1}^{\infty} g_n$ converges uniformly on D , we

must show (s_n) is uniformly Cauchy. Let $\epsilon > 0$

and let $x \in D$. Since $\sum_{n=1}^{\infty} M_n$ converges,

$\exists N > 0$ such that $\sum_{k=m+1}^n M_k < \epsilon$ when $n > m > N$.

Suppose $n > m > N$. Then

$$|s_n(x) - s_m(x)| = \left| \sum_{k=m+1}^n g_k(x) \right|$$

$$\leq \sum_{k=m+1}^n |g_k(x)|$$

$$\leq \sum_{k=m+1}^n M_k$$

$$< \epsilon.$$

Problem 1. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$. Use the Weierstrass M -test to prove that f is continuous on \mathbb{R} . You may assume without proof that $\cos x$ is continuous on \mathbb{R} .

Let $f_n(x) = \frac{1}{n^2} \cos nx$ and $M_n = \frac{1}{n^2} \quad \forall n \geq 1, x \in \mathbb{R}$.

Since $\sum_{n=1}^{\infty} M_n$ converges and $|f_n(x)| \leq M_n \quad \forall n \geq 1, x \in \mathbb{R}$,

$\sum_{n=1}^{\infty} f_n(x)$ converges uniformly by the Weierstrass M -test.

Since f_n is continuous, $\sum_{k=1}^n f_k$ is continuous $\forall n \geq 1$.

Therefore since the uniform limit of continuous

functions is continuous, $\sum_{n=1}^{\infty} f_n$ is continuous.

Problem 2. Let $0 < a < 1$. Use the Weierstrass M -test to prove that $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$ to $f(x) = 1/(1-x)$.

Let $f_n(x) = x^n$ and $M_n = a^n \quad \forall n \geq 0, x \in [-a, a]$.

Since $\sum_{n=0}^{\infty} M_n$ converges and $|f_n(x)| \leq M_n \quad \forall n \geq 0, x \in \mathbb{R}$,

$\sum_{n=0}^{\infty} f_n(x)$ converges uniformly by the Weierstrass M -test.

Note we proved earlier in the semester that

$$\sum_{n=0}^{\infty} f_n(x) = \frac{1}{1-x} .$$

Problem 3. In Homework 11 you are asked to prove the following theorem:

If (f_n) is a sequence of bounded functions on D and $f_n \rightarrow f$ uniformly on D then f is bounded on D .

State the contrapositive of this theorem.

If f is not bounded on D , then either $f_n \not\rightarrow f$ uniformly on D or f_n is not bounded for some $n \geq 1$.

Problem 4. Prove that $\sum_{n=0}^{\infty} x^n$ does not converge uniformly to $f(x) = 1/(1-x)$ on $(-1, 1)$.

Since f is not bounded on $(-1, 1)$ and

$f_n(x) = x^n$ is bounded on $(-1, 1)$ $\forall n \geq 0$,

$f_n \rightarrow f$ uniformly on $(-1, 1)$ by Problem 3.

Problem 5. The following function is known as the *Weierstrass function*. Let a, b be given constants such that $a \in (0, 1)$, b is a positive, odd integer, and $ab > 1 + 3\pi/2$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x).$$

a. Prove that f is continuous on \mathbb{R} .

Let $f_n(x) = a^n \cos(b^n \pi x)$ and $M_n = a^n \quad \forall x \in \mathbb{R}, n \geq 0$.

Since $\sum_{n=0}^{\infty} M_n$ converges and $|f_n(x)| \leq M_n \quad \forall n \geq 0, x \in \mathbb{R}$,

$\sum_{n=0}^{\infty} f_n(x)$ converges uniformly by the Weierstrass M-test.

Since f_n is continuous, $\sum_{k=0}^n f_k$ is continuous $\forall n \geq 1$.

Therefore since the uniform limit of continuous

functions is continuous, $\sum_{n=0}^{\infty} f_n$ is continuous.