

§ 27 Weierstrass Approximation Theorem

Useful identities Suppose $x \in [0, 1]$.

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = 1 \quad (\text{sum of binomial probabilities is } 1)$$

$$\sum_{k=0}^n k \binom{n}{k} x^k (1-x)^{n-k} = nx \quad (\text{mean of binomial distribution is } nx)$$

$$\sum_{k=0}^n (k - nx)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \quad (\text{variance})$$

$$\sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n}$$

These can all be proved using probability

theory, or induction, or brute force algebra.

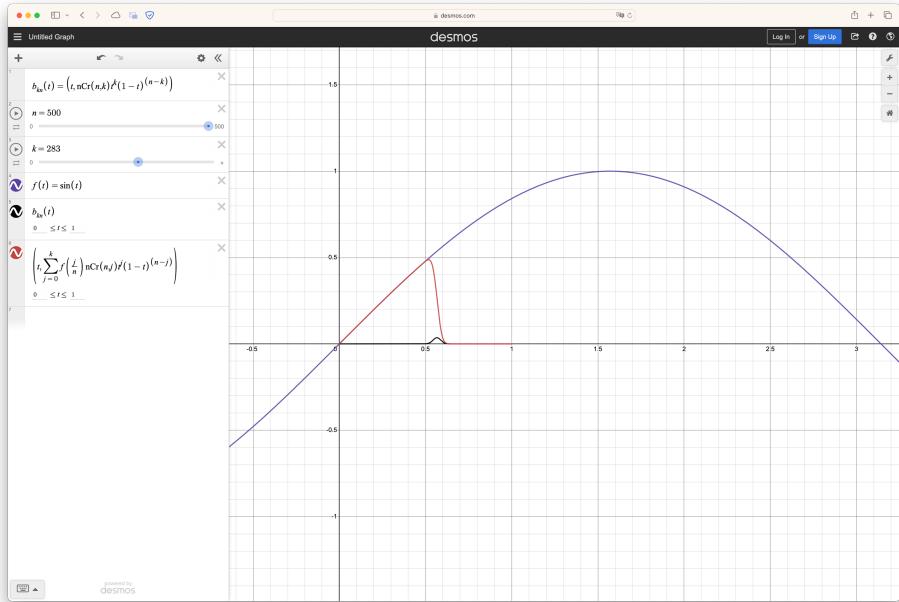
Important heuristic

For large n , given $0 \leq k \leq n$

- $\binom{n}{k} x^k (1-x)^{n-k}$ is largest when $x \approx \frac{k}{n}$
- $\binom{\frac{n}{k}}{k} x^k (1-x)^{n-k} \approx 0$ when x is not near $\frac{k}{n}$.

Intuition:

$$f(x) = f(x) \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \stackrel{?}{\approx} \sum_{k=0}^n f\left(\frac{n}{k}\right) \binom{n}{k} x^k (1-x)^{n-k}$$



Theorem (Weierstrass approximation) Let $f: [a,b] \rightarrow \mathbb{R}$

be a continuous function. There exists a sequence (p_n) of polynomials that converges uniformly on $[a,b]$ to f . In other words

$\forall \varepsilon > 0 \exists$ polynomial $p: [a,b] \rightarrow \mathbb{R}$ such that

$$|f(x) - p(x)| < \varepsilon \quad \forall x \in [a,b].$$

Proof We'll focus on the case when $[a,b]$ is $[0,1]$.

Let $\varepsilon > 0$ and let $x \in [0,1]$. Define $\forall n \geq 1$,

$$0 \leq k \leq n, \quad b_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{and let}$$

$$p_n(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) b_{k,n}(x).$$

Let $n \geq 1$. Observe that

$$\begin{aligned} |P_n(x) - f(x)| &= \left| \sum_{k=0}^n f\left(\frac{k}{n}\right) b_{k,n}(x) - \sum_{k=0}^n f(x) b_{k,n}(x) \right| \\ &= \left| \sum_{k=0}^n (f\left(\frac{k}{n}\right) - f(x)) b_{k,n}(x) \right| \\ &\leq \sum_{k=0}^n |f\left(\frac{k}{n}\right) - f(x)| b_{k,n}(x). \end{aligned}$$

Since f is uniformly continuous on $[0, 1]$,

$\exists \delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ for all

$x, y \in [0, 1]$ such that $|x - y| < \delta$. Let

$$A = \left\{ k : \left| \frac{k}{n} - x \right| < \delta \right\} \text{ and } B = \left\{ k : \left| \frac{k}{n} - x \right| \geq \delta \right\}.$$

Then

$$\begin{aligned} &\sum_{k=0}^n |f\left(\frac{k}{n}\right) - f(x)| b_{k,n}(x) \\ &= \sum_{k \in A} |f\left(\frac{k}{n}\right) - f(x)| b_{k,n}(x) + \sum_{k \in B} |f\left(\frac{k}{n}\right) - f(x)| b_{k,n}(x) \\ &< \frac{\varepsilon}{2} \sum_{k \in A} b_{k,n}(x) + \sum_{k \in B} |f\left(\frac{k}{n}\right) - f(x)| b_{k,n}(x) \\ &\leq \frac{\varepsilon}{2} + \sum_{k \in B} |f\left(\frac{k}{n}\right) - f(x)| b_{k,n}(x). \end{aligned}$$

Since f is continuous on $[0, 1]$, $\exists M > 0$
such that $|f(x)| \leq M \quad \forall x \in [0, 1]$ by the
Extreme Value Theorem. Therefore

$$\begin{aligned}
\sum_{k \in B} |f(\frac{k}{n}) - f(x)| b_{k,n}(x) &\leq 2M \sum_{k \in B} b_{k,n}(x) \\
&= \frac{2M}{\delta^2} \sum_{k \in B} \delta^2 b_{k,n}(x) \\
&\leq \frac{2M}{\delta^2} \sum_{k=0}^n (\frac{k}{n} - x)^2 b_{k,n}(x) \\
&\leq \frac{2M}{\delta^2} \sum_{k=0}^n \frac{x(1-x)}{n} b_{k,n}(x) \\
&= \frac{2M}{\delta^2} \cdot \frac{1}{n} \\
&\leq \frac{2M}{\delta^2} \cdot \frac{1}{n} \\
&= \frac{M}{2\delta} \cdot \frac{1}{n}
\end{aligned}$$

Let $N = \frac{M}{\delta\varepsilon}$ and suppose $n > N$. Then

$$\frac{M}{2\delta} \cdot \frac{1}{n} < \frac{M}{2\delta} \cdot \frac{1}{N} = \frac{\varepsilon}{2}.$$

Therefore, when $n > N$

$$|p_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$