Theorem Let (f_n) be a sequence of continuous functions that converges uniformly on D to f. Then f is continuous on D.

Proof Let $\varepsilon > 0$, $x_0 \in D$. We'll prove f is continuous at x_0 by proving $\exists\ \delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ when $x \in D$ and $0 < |x - x_0| < \delta$. Since $f_n \to f$ uniformly, $\exists\ N$ such that $|f_n(x) - f(x)| < \varepsilon/3$ for any x when $n > N$. Since f_{N+1} is continuous at x_0, $\exists\ \delta > 0$ such that $|f_{N+1}(x) - f_{N+1}(x_0)| < \varepsilon/3$ when $x \in D$, $0 < |x - x_0| < \delta$. We claim this is our desired δ.

Let $x \in D$ such that $0 < |x - x_0| < \delta$. Then

$$
\begin{aligned}
|f(x)-f(x_0)| &= |f(x)-f_{N+1}(x)+f_{N+1}(x)-f_{N+1}(x_0)+f_{N+1}(x_0)-f(x_0)| \\
&\leq |f(x)-f_{N+1}(x)| + |f_{N+1}(x)-f_{N+1}(x_0)| + |f_{N+1}(x_0)-f(x_0)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}
$$

__Remark__ This theorem is saying we can interchange order of limits __when__ convergence is uniform:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \lim_{n \to \infty} f_n(x)$$

$$\overset{(*)}{=} \lim_{n \to \infty} \lim_{x \to x_0} f_n(x)$$

$$= \lim_{n \to \infty} f_n(x_0)$$

$$= f(x_0)$$

The step $(*)$ only works when we have uniform convergence.

Question If $f_n \to f$ pointwise on $[a,b]$,

can we say $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$?

In general, no, as Problem 1 on worksheet will show

Theorem If (f_n) is a sequence of continuous functions on $[a,b]$ that converges uniformly to f on $[a,b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proved in worksheet.