§1.4 Consequences of Completeness.

Theorem (Archmidean Property)

(i) For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

(ii) For every $y \in (0,\infty)$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < y$.

Proof (i) Notice that this statement is equivalent to the statement "$\mathbb{N}$ is not bounded above."

Suppose by way of contradiction that $\mathbb{N}$ is bounded above. Then since $\mathbb{N} \neq \emptyset$, by the Axiom of Completeness, $\mathbb{N}$ has a least upper bound $\alpha = \sup \mathbb{N}$. Notice then that for each $\varepsilon > 0$, there exist $n \in \mathbb{N}$ such that $n > \alpha - \varepsilon$. So there exists $n_0 \in \mathbb{N}$ such that $n_0 > \alpha - 1$. However, $n_0 + 1 \in \mathbb{N}$ and $n_0 + 1 > \alpha$, which contradicts that α is an upper bound of $\mathbb{N}$.

(ii) Let $y \in (0,\infty)$. By (i), there exists $m \in \mathbb{N}$ such that $m > \frac{1}{y}$. Thus $\frac{1}{m} < y$.

Well-ordering principle of $\mathbb{N}$ Every nonempty subset of $\mathbb{N}$ has a minimum. (An axiom we assume about $\mathbb{N}$)

Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$) Let $a, b \in \mathbb{R}$ such that $a < b$. Then there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof We'll do the case when $a, b \in (0, \infty)$ and leave the other cases as an exercise.

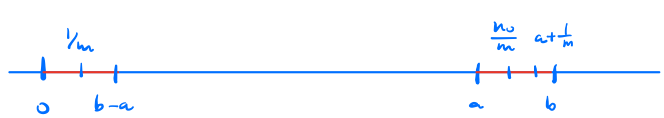
By the Archimedean property (ii) there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < b - a$. Consider the set

$$S = \left\{ n \in \mathbb{N} : \frac{n}{m} > a \right\}.$$

By the Archimedean property (i) S is nonempty.

By the well-ordering principle of $\mathbb{N}$, S has a minimum, $n_0 = \min S$. We claim $\frac{n_0}{m} < b$. Notice since $n_0 - 1 < n_0$ and $n_0 = \min S$, $\frac{n_0 - 1}{m} \le a$ which implies $\frac{n_0}{m} \le a + \frac{1}{m} < b$.

Therefore $a < \frac{n_0}{m} < b$, so $\exists\, q \in \mathbb{Q}$ s.t. $a < q < b$.



Problem 1. Let $S = \{1 - 1/n : n \in \mathbb{N}\}$. Use the Archimedean property to prove that $\sup S = 1$. To get you started, let me suggest two possible approaches, listed below. Please feel free to use either.

a. Prove that 1 is an upper bound of S and then prove that $1 \leq u$ for any upper bound u of S.

b. Use the lemma from last time which states that $\sup S = \alpha$ if and only if for every $\epsilon > 0$ there exists $x \in S$ such that $x > \alpha - \epsilon$.

Observe that $1 - \frac{1}{n} \leq 1$ for all $n \in \mathbb{Z}^+$. Thus 1 is an upper bound of S. Let u be an upper bound of S. We claim $u \geq 1$. Suppose not. Then $u < 1$. By the Archimedean property, there exists $m \in \mathbb{Z}^+$ such that $\frac{1}{m} < 1 - u$. But this implies $u < 1 - \frac{1}{m}$ which contradicts that u is an upper bound of S.

Problem 2. This problem outlines a proof of the Nested Interval Property, which is stated above.

a. Let $A = \{a_n : n \in \mathbb{N}\}$ be the collection of left endpoints of the closed intervals I_n.

   1. Explain why A is bounded above. Do this by listing out all upper bounds of A that you can identify.
   2. Explain why A has a supremum.

b. Let $x = \sup A$.

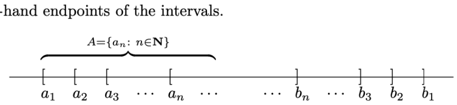
   1. Explain why $a_n \leq x$ for all $n \in \mathbb{N}$.
   2. Explain why $x \leq b_n$ for all $n \in \mathbb{N}$.

c. Explain why $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. In order to show that $\bigcap_{n=1}^{\infty} I_n$ is not empty, we are going to use the Axiom of Completeness (AoC) to produce a single real number x satisfying $x \in I_n$ for every $n \in \mathbf{N}$. Now, AoC is a statement about bounded sets, and the one we want to consider is the set

$$A = \{a_n : n \in \mathbf{N}\}$$

of left-hand endpoints of the intervals.



Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we are justified in setting

$$x = \sup A.$$

Now, consider a particular $I_n = [a_n, b_n]$. Because x is an upper bound for A, we have $a_n \leq x$. The fact that each b_n is an upper bound for A and that x is the least upper bound implies $x \leq b_n$.

Altogether then, we have $a_n \leq x \leq b_n$, which means $x \in I_n$ for every choice of $n \in \mathbf{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is not empty. $\square$