§7.8 Introduction to sequences

Def A sequence is an ordered infinite list of numbers:

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \dots)$$

Note we can view a sequence as a function $a : \mathbb{N} \to \mathbb{R}$ where $a(n) = a_n$.

Goal Study long-term behavior of sequences.

Examples Use calculus intuition to state the limit of the following sequences.

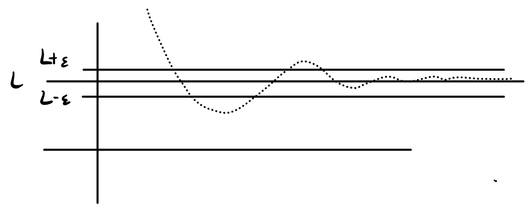
1. $a_n = \frac{1}{n^2}$ $\quad 0$
2. $a_n = \frac{(-1)^n}{\sqrt{n}}$ $\quad 0$
3. $a_n = \frac{5n^2+3}{4n^2-1}$ $\quad \frac{5}{4}$
4. $a_n = \sin\left(\frac{n\pi}{2}\right)$ $\quad$ DNE

Def Let $(a_n) \subseteq \mathbb{R}$ be a given sequence and let $L \in \mathbb{R}$. Then (a_n) converges to L if $\forall\, \varepsilon > 0$ $\exists\, N > 0$ such that when $n > N$,

$$|a_n - L| < \varepsilon.$$

In simpler terms (a_n) converges to L if terms a_n eventually get ε-close to L for arbitrary choice of $\varepsilon > 0$.

Visual



for large enough n, $|a_n - L| < \varepsilon$.

Example Give an ε-N proof that $a_n = \frac{1}{n^2}$ converges to $L = 0$.

Proof Let $\varepsilon > 0$ and define $N = \frac{1}{\sqrt{\varepsilon}}$.

Suppose $n > N$. Then

$$
\begin{aligned}
|a_n - L| &= \left|\frac{1}{n^2}\right| \\
&= \frac{1}{n^2} \\
&< \frac{1}{N^2} \\
&= \varepsilon
\end{aligned}
$$

Example Give an ε-N proof that $\lim_{n\to\infty} 5 + \frac{(-1)^n}{\sqrt{n}} = 5$.

Proof Let $\varepsilon > 0$ and define $a_n = 5 + \frac{(-1)^n}{\sqrt{n}}$ for $n \geq 1$, $L = 5$, and $\underline{N = \frac{1}{\varepsilon^2}}$

Suppose $n > N$. Then

$$
\begin{aligned}
|a_n - L| &= \left| \frac{(-1)^n}{\sqrt{n}} \right| \\
&= \frac{1}{\sqrt{n}} \\
&< \frac{1}{\sqrt{N}} \\
&= \varepsilon.
\end{aligned}
$$

Problem 1. For each sequence (a_n) and given value L, we will prove that $\lim_{n\to\infty} a_n = L$. The structure of the proofs was outlined in class: we want to find N so that $|a_n - L| < \epsilon$ when $n > N$.

a. $a_n = 1 + 2/n, L = 1$

b. $a_n = \sin(n)/n^3, L = 0$

(a) Proof Let $\varepsilon > 0$. Define $N = 2/\varepsilon$.

Observe that when $n > N$,

$$|a_n - L| = \left|1 + \frac{2}{n} - 1\right| = \left|\frac{2}{n}\right| = \frac{2}{n} < \frac{2}{N} = \frac{2}{2/\varepsilon} = \varepsilon$$

(find N in this step by solving $\frac{2}{N} = \varepsilon$) $N = \frac{2}{\varepsilon}$

(b) Proof Let $\varepsilon > 0$. Define $N = (1/\varepsilon)^{1/3}$.

Observe that when $n > N$,

$$|a_n - L| = \left|\frac{\sin(n)}{n^3}\right| \le \frac{1}{n^3} < \frac{1}{N^3} = \varepsilon$$

Problem 2. Sometimes the algebra is a little more involved and there's more scratch work to do. We also might not be told the value of the limit and we must use our intuition and skills from calculus. Use the same technique and structure as the previous proofs to prove that the following sequence converges:

$$a_n = \frac{3n+1}{7n-4}.$$

Proof We claim $\lim_{n \to \infty} a_n = \frac{3}{7}$. Let $\varepsilon > 0$, $L = \frac{3}{7}$ and define $N = \frac{1}{7}\left(\frac{19}{7\varepsilon} + 4\right)$. Suppose $n > N$.

Then

$$|a_n - L| = \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right|$$

$$= \left| \frac{7(3n+1) - 3(7n-4)}{7(7n-4)} \right|$$

$$= \left| \frac{19}{7(7n-4)} \right|$$

$$= \frac{19}{7(7n-4)}$$

$$< \frac{19}{7(7N-4)}$$

$$= \varepsilon$$