

## §8 More basics of $\varepsilon$ -N proofs

Example Let  $a_n = \frac{4n^3 + 3n}{n^3 - 6}$  and  $L = 4$ .

Prove  $\lim_{n \rightarrow \infty} a_n = L$ .

Proof Let  $\varepsilon > 0$ . Define  $N = \max \left\{ 12^{\frac{1}{3}}, \left( \frac{54}{\varepsilon} \right)^{\frac{1}{2}} \right\}$

Suppose  $n > N$ . Then

$$\begin{aligned} |a_n - L| &= \left| \frac{\frac{4n^3 + 3n}{n^3 - 6} - 4}{1} \right| \\ &= \left| \frac{4n^3 + 3n - 4(n^3 - 6)}{n^3 - 6} \right| \\ &= \left| \frac{3n + 24}{n^3 - 6} \right| \\ &= \frac{3n + 24}{n^3 - 6} \quad \text{since } n > 12^{\frac{1}{3}} \end{aligned}$$

Pausing proof for comments:

(1) for large enough  $n$ , there should be some

$C$  so that  $\frac{3n+24}{n^3-6} \leq \frac{C}{n^2}$

(2) for large enough  $n$ ,  $\frac{C}{n^2} < \varepsilon$

Regarding ① :

$$\bullet \quad 3n+24 \leq 3n+24n = 27n \text{ for all } n \geq 1$$

$$\bullet \quad n^3 - 6 \geq \frac{1}{2}n^3 \text{ eventually :}$$

$$\begin{aligned} n^3 - 6 \geq \frac{1}{2}n^3 &\Leftrightarrow \frac{1}{2}n^3 \geq 6 \\ &\Leftrightarrow n^3 \geq 12 \\ &\Leftrightarrow n \geq 12^{\frac{1}{3}} \end{aligned}$$

$$\text{When } n > 12^{\frac{1}{3}}, \quad \frac{3n+24}{n^3 - 6} \leq \frac{27n}{\frac{1}{2}n^3} = \frac{54}{n^2}$$

Regarding ② :

$$\text{When } n > \sqrt{\frac{54}{\varepsilon}}, \quad \frac{54}{n^2} < \varepsilon$$

Resuming proof :

$$\leq \frac{3n+24n}{\frac{1}{2}n^3} \quad \text{since } n > 12^{\frac{1}{3}}$$

$$= \frac{54}{n^2}$$

$$< \varepsilon \quad \text{since } n > \sqrt{\frac{54}{\varepsilon}}$$

Note a proof in your homework should include a discussion like the red comments above so that the details are clear.

Example Prove that  $a_n = (-1)^n$  diverges.

Proof Suppose  $a_n$  converges to some value  $L$ . Then  $\forall \varepsilon > 0 \exists N$  so that  $|a_n - L| < \varepsilon$  when  $n > N$ . In particular, for  $\varepsilon_0 = 0.1$ , there exists  $N_0$  so that for all  $n > N_0$ ,

$$|(-1)^n - L| < 0.1 \quad \text{and} \quad |(-1)^{n+1} - L| < 0.1.$$

Notice that  $|(-1)^n - (-1)^{n+1}| = 2$  for all  $n \geq 1$ .

Suppose  $n > N_0$ . Then

$$\begin{aligned} 2 &= |(-1)^n - (-1)^{n+1}| \\ &\leq |(-1)^n - L| + |L - (-1)^{n+1}| \\ &< 0.1 + 0.1 = 0.2, \end{aligned}$$

a contradiction.

**Problem 1.** The examples from our first day of sequences were hopefully straightforward, at least up to a little bit of algebra to be done. The examples in this problem ask you do similar proofs, but finding  $N$  takes a little more work. These are more like Example 3 in Section 8.

a.  $a_n = \frac{n^2+3}{n^2-3}, L = 1$

b.  $a_n = \frac{n^2-3n+2}{n^2+3}, L = 1$

(a) Let  $\varepsilon > 0$  and define  $N = \max\{\sqrt{6}, \sqrt{12/\varepsilon}\}$ .

Suppose  $n > N$  and observe that

$$\begin{aligned} |a_n - L| &= \left| \frac{n^2+3}{n^2-3} - 1 \right| \\ &= \left| \frac{n^2+3 - (n^2-3)}{n^2-3} \right| \\ &= \left| \frac{6}{n^2-3} \right| \\ &= \frac{6}{n^2-3} \quad (1) \\ &\leq \frac{6}{\frac{1}{2}n^2} \quad (2) \\ &= \frac{12}{n^2} \\ &< \varepsilon. \quad (3) \end{aligned}$$

Note (1) holds since  $n > \sqrt{6} \Rightarrow n^2 - 3 > 0$ ,

(2) holds since  $n^2 - 3 \geq \frac{1}{2}n^2 \Leftrightarrow n^2 \geq 6 \Leftrightarrow n \geq \sqrt{6}$

and (3) hold since  $n > \sqrt{12/\varepsilon}$ .

⑥ Let  $\varepsilon > 0$  and define  $N = \frac{\varepsilon}{4}$ .

Suppose  $n > N$  and observe that

$$\begin{aligned} |a_n - L| &= \left| \frac{n^2 - 3n + 2}{n^2 + 3} - 1 \right| \\ &= \left| \frac{n^2 - 3n + 2 - (n^2 + 3)}{n^2 + 3} \right| \\ &= \left| \frac{-3n - 1}{n^2 + 3} \right| \\ &= \frac{3n + 1}{n^2 + 3} \\ &\leq \frac{3n + n}{n^2} \\ &= \frac{4}{n} \\ &< \varepsilon \end{aligned}$$

**Problem 2.** Showing a sequence diverges often times requires a proof by contradiction. Try emulating the proof that  $a_n = (-1)^n$  diverges (Example 4 in Section 8) to show that  $a_n = \sin(n\pi/2)$  diverges.

Proof Suppose  $a_n$  converges to some value  $L$ . Then  $\forall \varepsilon > 0 \exists N$  so that  $|a_n - L| < \varepsilon$  when  $n > N$ . In particular, for  $\varepsilon_0 = 0.1$ , there exists  $N_0$  so that for all  $n > N_0$ ,

$$|a_n - L| < 0.1 \text{ and } |a_{n+2} - L| < 0.1.$$

Notice that  $|a_n - a_{n+2}| = 2$  for all  $n \geq 1$  odd.

Suppose  $n > N_0$  and  $n$  odd. Then

$$\begin{aligned} 2 &= |a_n - a_{n+2}| \\ &\leq |a_n - L| + |L - a_{n+2}| \\ &< 0.1 + 0.1 = 0.2, \end{aligned}$$

a contradiction.