§ 8, 9 Limit Theorems

Lemma 1 $x = y$ if and only if $\forall\, \varepsilon > 0$, $|x - y| < \varepsilon$.

Theorem Let (a_n) be a convergent sequence. Then its limit is unique. In other words, if $\lim\limits_{n\to\infty} a_n = L_1$ and $\lim\limits_{n\to\infty} a_n = L_2$, then $L_1 = L_2$.

Proof Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} a_n = L_1$ there exists N_1 so that $|a_n - L_1| < \varepsilon/2$ when $n > N_1$. Since $\lim\limits_{n\to\infty} a_n = L_2$, there exists N_2 so that $|a_n - L_2| < \varepsilon/2$ when $n > N_2$. Suppose $n > \max\{N_1, N_2\}$.

Then

$$
\begin{aligned}
|L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \\
&\le |L_1 - a_n| + |a_n - L_2| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon .
\end{aligned}
$$

Therefore $L_1 = L_2$ by Lemma 1.

Lemma 2 $x \leq y$ if and only if $\forall\, \varepsilon > 0,\ x < y + \varepsilon$.

Theorem Let $(a_n) \subseteq \mathbb{R}$ be a convergent sequence with $\lim_{n\to\infty} a_n = L$. If $a_n < c$ for all but finitely many n, then $L \leq c$.

Proof Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there exists N_1 so that $|a_n - L| < \varepsilon$ when $n > N_1$.

Therefore, $L - \varepsilon < a_n < L + \varepsilon$ when $n > N_1$.

Since $a_n < c$ for all but finitely many n, there exists N_2 so that $a_n < c$ when $n > N_2$.

Suppose $n > \max\{N_1, N_2\}$. Then

$$\begin{aligned} L &< a_n + \varepsilon && \text{since } n > N_1 \\ &< c + \varepsilon && \text{since } n > N_2. \end{aligned}$$

Therefore $L \leq c$ by Lemma 2.

Problem 1. Let (a_n) be a convergent sequence and let $L = \lim_{n\to\infty} a_n$. Suppose $k \neq 0$. Prove that $\lim ka_n = kL$. (*Remark: this fact is of course true when $k = 0$ also but you need not consider that case here.*)

Proof Let $\varepsilon > 0$. There exists N so that $|a_n - L| < \frac{\varepsilon}{|k|}$ when $n > N$. Suppose $n > N$ and observe that

$$|ka_n - kL| = |k||a_n - L| < |k| \frac{\varepsilon}{|k|} = \varepsilon.$$

Problem 2. Let (a_n) and (b_n) be convergent sequences and let $L_1 = \lim_{n\to\infty} a_n$ and $L_2 = \lim_{n\to\infty} b_n$. Prove that $\lim_{n\to\infty}(a_n + b_n) = L_1 + L_2$.

Proof Let $\varepsilon > 0$. There exist N_1, N_2 so that $|a_n - L_1| < \varepsilon/2$ when $n > N_1$ and $|b_n - L_2| < \varepsilon/2$ when $n > N_2$. Let $N = \max\{N_1, N_2\}$. Suppose $n > N$ and observe that

$$|(a_n + b_n) - (L_1 + L_2)| \leq |a_n - L_1| + |b_n - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Problem 3. Let (a_n) be a convergent sequence and let $L = \lim_{n\to\infty} a_n$. Suppose $a_n > c$ for all but finitely many $n \geq 1$. Prove that $L \geq c$.

Proof. Let $\varepsilon > 0$. There exists N_1 so that $|a_n - L| < \varepsilon$ when $n > N_1$. Thus $a_n < L + \varepsilon$ when $n > N_1$.

Moreover there exists N_2 so that $c < a_n$ for all $n > N_2$. Suppose $n > \max\{N_1, N_2\}$. Observe that

$$c < a_n < L + \varepsilon.$$

Therefore $c \leq L$.

Problem 4. Give an example of a convergent sequence (a_n) where $a_n > 0$ for all but finitely many $n \geq 1$ whose limit L fails to satisfy the strict inequality $L > 0$. This example is meant to explain why we *need* the non-strict inequality in the theorem you've proved in Problem 3.

$$a_n = \frac{1}{n}.$$