§9 More Limit Theorems

Theorem Let (a_n) be a convergent sequence. Then (a_n) is bounded. That is, there exists $M>0$ such that $|a_n|<M$ for all $n\geq 1$.

Proof Let $\varepsilon>0$ and let $L=\lim\limits_{n\to\infty} a_n$. There exists N such that $|a_n-L|<\varepsilon$ when $n>N$.

Therefore, when $n>N$,

$$
\begin{aligned}
|a_n| &= |a_n - L + L| \\
&\leq |a_n - L| + |L| \\
&< \varepsilon + |L|.
\end{aligned}
$$

Let $M=\max\{|a_1|, |a_2|, \ldots, |a_N|, \varepsilon+|L|\}$.

Then $|a_n|\leq M$ for all $n\geq 1$.

Theorem (Algebraic Limit Theorems) Let (a_n) and (b_n) be convergent sequences with $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$. Let $k \in \mathbb{R}$. Then

① $\lim_{n\to\infty} ka_n = kL_1$

② $\lim_{n\to\infty} (a_n+b_n) = L_1 + L_2$

Proved in previous worksheet

③ $\lim_{n\to\infty} a_n b_n = L_1 L_2$

④ if $L_2 \neq 0$ and $b_n \neq 0$ $\forall n \geq 1$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L_1}{L_2}$.

Proof of ③ Let $\varepsilon > 0$. Since (a_n) converges, it is bounded and there exists $M > 0$ so that $|a_n| \leq M$ for all $n \geq 1$. Also, there exists N_1 so that $|a_n - L_1| < \frac{\varepsilon}{2|L_2|}$ for all $n > N_1$, and there exists N_2 so that $|b_n - L_2| < \frac{\varepsilon}{2M}$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$. Suppose $n > N$. Observe that

$$\begin{aligned}
|a_n b_n - L_1 L_2| &= |a_n b_n - a_n L_2 + a_n L_2 - L_1 L_2| \\
&\leq |a_n b_n - a_n L_2| + |a_n L_2 - L_1 L_2| \\
&= |a_n||b_n - L_2| + |L_2||a_n - L_1| \\
&\leq M|b_n - L_2| + |L_2||a_n - L_1| \\
&< M\left(\frac{\varepsilon}{2M}\right) + |L_2|\left(\frac{\varepsilon}{2|L_2|}\right) \\
&= \varepsilon
\end{aligned}$$

Theorem (Order Limit Theorems) Let (a_n) and (b_n) be convergent sequences with $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$. Then

1. if $a_n > c$ for all but finitely many n, $L_1 \geq c$.
2. if $a_n < c$ for all but finitely many n, $L_2 \leq c$.
3. if $a_n < b_n$ for all but finitely many n, $L_1 \leq L_2$

Proof of ③

Let $c_n = a_n - b_n$. By Algebraic Limit Theorems, (c_n) converges and $\lim_{n\to\infty} c_n = L_1 - L_2$

Note $c_n < 0$ for all but finitely many n.

By Order Limit Theorem ②, $L_1 - L_2 \leq 0$. Therefore $L_1 \leq L_2$.