

## § 10 Monotone Sequences

Def A sequence is monotone if either

- ① it's increasing :  $a_n \leq a_{n+1}$  for all  $n \geq 1$   
or ② it's decreasing :  $a_n \geq a_{n+1}$  for all  $n \geq 1$ .

Theorem (Monotone Convergence Theorem)

- ① If  $(a_n)$  is bounded above and increasing  
then it converges.



- ② If  $(a_n)$  is bounded below and decreasing  
then it converges.

- ③ If  $(a_n)$  is bounded and monotone, then  
it converges.

We'll prove ①, leave ② for homework

Discussion (a) What do you think limit  $L$  is?

(b) Let  $\varepsilon > 0$ . What can you say about  $L - \varepsilon$ ?

Proof of ① We claim  $(a_n)$  converges to

$L = \sup \{a_n : n \geq 1\}$ . Let  $\varepsilon > 0$ . Notice  
that there exists an element  $a_N \in \{a_n : n \geq 1\}$   
so that  $L - \varepsilon < a_N$  by the Supremum  
Lemma. That is there exists  $N > 0$  so that

$$L - \varepsilon < a_N$$

Suppose  $n > N$ . Then

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$$

and so  $|a_n - L| < \varepsilon$  when  $n > N$ .

Example Let  $a_{n+1} = \frac{a_n^2 + 5}{2a_n}$ ,  $a_1 = 5$ .

(1) Assume  $(a_n)$  converges. Find its limit  $L$ .

(2) Prove  $a_n > L$  for all  $n \geq 1$  by induction.

(3) Prove  $(a_n)$  is decreasing

(1) If  $(a_n)$  converges and  $\lim_{n \rightarrow \infty} a_n = L$ , then

$$L = \frac{L^2 + 5}{2L}$$

$$\Leftrightarrow 2L^2 = L^2 + 5$$

$$\Leftrightarrow L^2 = 5 \Leftrightarrow L = \sqrt{5}.$$

(2) Base case  $a_1 = 5 > \sqrt{5}$ .

Induction step Assume  $a_n > \sqrt{5}$ .

Scratch work  $a_{n+1} > \sqrt{5}$

$$\Leftrightarrow \frac{a_n^2 + 5}{2a_n} > \sqrt{5}$$

$$\Leftrightarrow a_n^2 + 5 > 2\sqrt{5} a_n$$

$$\Leftrightarrow a_n^2 - 2\sqrt{5} a_n + 5 > 0$$

$$\Leftrightarrow (a_n - \sqrt{5})^2 > 0$$

$$\Leftrightarrow a_n > \sqrt{5}$$

Actual proof should start at bottom  
and go up.

(c) Scratch work

$$\begin{aligned}
 & a_{n+1} \leq a_n \\
 \Leftrightarrow & \frac{a_n^2 + 5}{2a_n} \leq a_n \\
 \Leftrightarrow & a_n^2 + 5 \leq 2a_n \\
 \Leftrightarrow & a_n^2 \geq 5 \\
 \Leftrightarrow & a_n \geq \sqrt{5} \quad \text{Actual proof goes} \\
 & \text{from bottom to top.}
 \end{aligned}$$

**Problem 1.** Consider the sequence given by  $a_1 = 9$  and  $a_{n+1} = \frac{1}{2} \left( a_n + \frac{9}{a_n} \right)$  for all  $n \geq 1$ .

- Suppose the sequence converges to a value  $L$ . Find  $L$ .
- Make a conjecture for whether the sequence is increasing or decreasing.
- If you believe the sequence is increasing, make a conjecture for an upper bound for the sequence. If you believe it is decreasing, make a conjecture for a lower bound for the sequence.
- Prove your conjectured bound using induction.
- Prove your increasing/decreasing conjecture.
- Make a conclusion.

(a)  $L = \frac{1}{2} \left( L + \frac{9}{L} \right)$

$$\begin{aligned}
 \Leftrightarrow & 2L^2 = L^2 + 9 \\
 \Leftrightarrow & L = \pm 3 \quad (\text{but only } L=3 \text{ is reasonable} \\
 & \text{since } a_n \geq 0 \text{ for all } n \geq 1).
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad a_1 &= 9, \quad a_2 = \frac{1}{2} (9+1) = 5, \quad a_3 = \frac{1}{2} \left( 5 + \frac{9}{5} \right) \\
 &= \frac{17}{5} = 3.4
 \end{aligned}$$

Decreasing

④ Claim  $a_n > 3$  for all  $n \geq 1$

Proof Base case:  $a_1 = 9 > 3$ .

Induction step: Assume  $a_n > 3$ .

Then  $a_{n+1} > 3$

$$\begin{aligned}\Leftrightarrow \frac{1}{2} \left( a_n + \frac{9}{a_n} \right) &> 3 \\ \Leftrightarrow a_n^2 + 9 &> 6a_n \\ \Leftrightarrow a_n^2 - 6a_n + 9 &> 0 \\ \Leftrightarrow (a_n - 3)^2 &> 0 \\ \Leftrightarrow a_n > 3 &\quad (\text{induction hypothesis})\end{aligned}$$

⑤ Claim  $(a_n)$  decreasing

Proof  $a_{n+1} \leq a_n$

$$\begin{aligned}\Leftrightarrow \frac{1}{2} \left( a_n + \frac{9}{a_n} \right) &\leq a_n \\ \Leftrightarrow a_n^2 + 9 &\leq 2a_n^2 \\ \Leftrightarrow 9 &\leq a_n^2 \\ \Leftrightarrow 3 &\leq a_n\end{aligned}$$

⑥ By monotone convergence theorem  $(a_n)$

converges. It converges to 3.