Math 301 — Subsequential limits and limsup, liminf

Summary. The notions of subsequential limits, accumulation points, limit supremum, and limit infimum are all tied together. Subsequential limits and accumulation points are synonymous notions, which we think of as the points that a sequence frequently gets nearby to. As we'll show below, the limit supremum and limit infimum are the supremum and infimum of the set of subsequential limits.

Problem 1. We've proven that for a sequence (a_n), $\limsup a_n$ is a subsequential limit and a similar proof can be given to show that $\liminf a_n$ is too. We'd like to prove that if S is the set of subsequential limits,

a. $\limsup a_n = \sup S$

b. $\liminf a_n = \inf S$

We'll prove the first equality above and the second one will be similar. Try filling in the blanks in the following proof.

Proof. Our aim is to show that $\limsup a_n = \sup S$, which we'll do by proving

$$\limsup a_n \leq \sup S \quad \text{and} \quad \sup S \leq \limsup a_n.$$

The work in proving the first inequality above is already done because BLANK1. Now, to prove the second inequality, we will use the fact that $\sup S$ is the least upper bound of S and show that $\limsup a_n$ is an BLANK2 of the set BLANK2.

Let $t \in S$. This means there exists a subsequence (a_{n_k}) such that BLANK3. Now, we aim to show that $t \leq \limsup a_n$. We begin by claiming that

$$\limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$

Once this claim is proved, our proof will be done since BLANK4. To prove our claim, notice that for any $N \geq 1$

$$\{a_{n_k} : k > N\} \subseteq \{a_n : n > N\}.$$

Therefore, BLANK4. $\square$

Problem 2. Consider the following sequences.

$$\begin{aligned}
(x_n) &= 5(-1)^n + \frac{1}{n} \\
(y_n) &= \cos\left(\frac{n\pi}{3}\right) \\
(z_n) &= (1, 0, -2, 0, 3, 0, -4, 0, 5, 0, -6, \ldots)
\end{aligned}$$

a. For each sequence, find the set of accumulation points. Include $\pm\infty$ as possible accumulation points.

b. Find the limsup and liminf of each sequence.