## Math 301 — Exam 2 review guide

Your second exam will have two parts, an in-class portion on April 11 and a take-home portion due April 14. The in-class portion will contain about 5 problems, some with multiple parts. You should expect to see a question asking you to state some definitions or theorems, a question asking you to prove a statement that was proved in class, and a few questions in the style similar to worksheets, homework, and quizzes. You should expect to write 2-3 proofs. It will cover material from Homework 5 to Homework 7. In the textbook, this is material spanning Sections 2.6-4.2, excluding sections not covered in lecture. I have outlined some important definitions, theorems, and general topics below. Also, the problems below give you a sampling of some problems like those that will appear on the exam, but it's not necessarily comprehensive, so make sure to review old homework, quizzes, worksheets, and lecture notes.

## Definitions and theorems

While not necessarily comprehensive, here is an absolutely-must-know list of definitions.

- Cauchy sequence, partial sum sequence of a series, convergent series, divergent series
- open set, limit point of a set, isolated point of a set, closed set, closure of a set, compact set
- functional limit

Similarly, here is an absolutely-must-know list of statements you should know.

- Cauchy criterion for sequences, Cauchy criterion for series, Algebraic Limit Theorem for Series, Comparison Test, Absolute Convergence Test, Alternating Series Test
- arbitrary unions and finite intersections of open sets are open, arbitrary intersections and finite unions of closed sets are open, sequential characterization of limit points (Theorem 3.2.5), sequential characterization of closed sets (Theorem 3.2.8), Heine-Borel Theorem (Theorem 3.3.4)
- sequential characterization of functional limit (Theorem 4.2.3), Algebraic Limit Theorem for functional limits, Divergence Criterion for functional limits

## Proofs and logic

You should know the following about proofs and logic. You should know how to:

- prove a given series converges with the Comparison Test
- prove a given set is open, not open, closed, not closed, compact, or not compact
- prove a given point is a limit point of a given set
- prove a given functional limit exists using the  $\epsilon$ - $\delta$  definition
- prove a given functional limit does not exist using sequences

## Sample problems

These problems are not comprehensive and there are more than you will see on the exam itself, but will give you an idea of the kinds of questions to expect.

**Problem 1.** Please state whether the following statements are true or false. If a statement is true, explain why. If a statement is false, give a counterexample and modify the statement slightly to make it true.

- a. If the series  $\sum a_n$  converges, then  $(a_n)$  is a Cauchy sequence.
- b. If  $A \subseteq \mathbb{R}$  is a given set and there exists a sequence  $(a_n) \subseteq A$  such that  $a_n \to x$ , then x is a limit point of A.
- c. If  $A_1, A_2, \ldots$  are open sets, then  $\bigcap_{n=1}^{\infty} A_n$  is an open set.
- d. If x is an isolated point of A then  $x \in \overline{A}$ .
- e. If K is a compact set, then every sequence  $(x_n) \subseteq K$  converges.

f. Let 
$$f(x) = \begin{cases} |x| & x \neq 0\\ 1 & x = 0 \end{cases}$$
. Then  $\lim_{x \to 0} f(x)$  does not exist.

Problem 2. Please do the following proof problems.

- a. Prove that if  $\sum_{n=1}^{\infty} a_n$  converges absolutely and  $(b_n)$  is a subsequence of  $(a_n)$  then  $\sum_{n=1}^{\infty} b_n$  converges absolutely.
- b. Let  $a, b \in \mathbb{R}$  be such that a < b. Prove from definitions and first principles that the set (a, b] is neither open nor closed.
- c. Prove that the union of a finite collection of compact sets is compact.
- d. Prove that the set  $A = \left\{\sum_{k=1}^{n} 1/k^2 : n \in \mathbb{N}\right\}$  is not compact using the fact that the series  $\sum_{k=1}^{\infty} 1/k^2$  converges to an irrational number  $\alpha$ .
- e. Give an  $\epsilon$ - $\delta$  proof showing that  $\lim_{x\to 3} x^3 = 27$ . Note that  $a^3 b^3 = (a-b)(a^2 + ab + b^2)$ .

f. Let 
$$f(x) = \begin{cases} x & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$
. Prove that  $\lim_{x \to \sqrt{2}} f(x)$  does not exist.