Math 301 — Uniform continuity

Summary. A function is continuous on a domain D if it's continuous at every point in the domain. When we prove a function is continuous at a given point using an ϵ-δ proof, our choice of δ might depend on both ϵ and the domain point we're considering. However, when our choice of δ can be chosen independently of the domain point, the function is called *uniformly continuous.*

Problem 1. Try proving the following functions $f : D \to \mathbb{R}$ are uniformly continuous on the given domain D by giving an ϵ-δ proof where δ depends only on ϵ.

a. $f(x) = x^2$, $D = [-4, 3]$.

b. $f(x) = 1/x$, $D = (1, 2)$.

c. $f(x) = 1/(x-3)$, $D = (4, \infty)$.

Problem 2. Suppose we wish to have a theorem that says something like, "If f is continuous on D, then f is uniformly continuous on D." What kind of set could D be? An open interval? A closed interval? A bounded set? Think about examples and make a conjecture. For a challenge try proving your conjecture using a proof by contradiction and the Bolzano Weierstrass theorem.

(a) Let $\varepsilon > 0$ and define $\delta = \frac{\varepsilon}{8}$.

Suppose $x, y \in [-4, 3]$ and $|x - y| < \delta$.

Then $|f(x) - f(y)| = |x^2 - y^2|$

$= |x - y||x + y|$

$< \delta |x + y|$

$$\le \delta\,(|x| + |y|)$$

$$\le 8\delta$$

$$= \varepsilon$$

ⓑ Let $\varepsilon > 0$ and define $\delta = \varepsilon$

Suppose $x, y \in (1,2)$ and $|x-y| < \delta$.

Then $$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right|$$

$$= \left|\frac{y-x}{xy}\right|$$

$$< \frac{\delta}{|x||y|}$$

$$< \delta$$

$$= \varepsilon.$$

ⓒ Let $\varepsilon > 0$ and define $\delta = \varepsilon$

Suppose $x, y \in (4, \infty)$ and $|x-y| < \delta$.

Then

$$
\begin{aligned}
|f(x) - f(y)| &= \left| \frac{1}{x-3} - \frac{1}{y-3} \right| \\
&= \left| \frac{y-3-(x-3)}{(x-3)(y-3)} \right| \\
&= \left| \frac{y-x}{(x-3)(y-3)} \right| \\
&< \frac{\delta}{|x-3||y-3|} \\
&< \delta \\
&= \varepsilon.
\end{aligned}
$$