Math 301 — Uniform continuity

Summary. A function is continuous on a domain D if it's continuous at every point in the domain. When we prove a function is continuous at a given point using an ϵ-δ proof, our choice of δ might depend on both ϵ and the domain point we're considering. However, when our choice of δ can be chosen independently of the domain point, the function is called *uniformly continuous.*

Problem 1. Try proving the following functions $f : D \to \mathbb{R}$ fail to be uniformly continuous on the given domain D.

a. $f(x) = 1/x$, $D = (0, 1)$.

b. $f(x) = x^2$, $D = \mathbb{R}$.

(a) We'll show $\exists$ $\varepsilon > 0$ and sequences (x_n) and (y_n) such that

$$\lim |x_n - y_n| = 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \varepsilon$$

for all n.

Consider $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$, $\varepsilon = 1$.

Then $|f(x_n) - f(y_n)| = |n - 2n| = n \geq \varepsilon$ for all n

and $\lim |x_n - y_n| = \lim \left| \frac{1}{n} - \frac{1}{2n} \right| = 0.$

(b) We'll show $\exists$ $\varepsilon > 0$ and sequences (x_n) and (y_n) such that

$$\lim |x_n - y_n| = 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \varepsilon$$

for all n.

Consider $x_n = n$, $y_n = n + \frac{1}{n}$, $\varepsilon = 2$

Then
$$\begin{aligned} |f(x_n) - f(y_n)| &= \left| n^2 - (n + \tfrac{1}{n})^2 \right| \\ &= \left| n^2 - (n^2 + 2 + \tfrac{1}{n^2}) \right| \\ &= 2 + \frac{1}{n^2} \\ &\geq 2 = \varepsilon \quad \text{for all } n \end{aligned}$$

and
$$\begin{aligned} \lim |x_n - y_n| &= \lim \left| n - (n + \tfrac{1}{n}) \right| \\ &= \lim \frac{1}{n} = 0. \end{aligned}$$