Math 301 — The fundamental theorem of calculus

Problem 1. Let $f : I \to \mathbb{R}$ be a given function. Prove that if $F, G : I \to \mathbb{R}$ are antiderivatives of f then there exists $C \in \mathbb{R}$ such that $F(x) = G(x) + C$ for all $x \in I$.

Problem 2. Let $f : I \to \mathbb{R}$ be a continuous function and let $F : I \to \mathbb{R}$ be an antiderivative of f. Prove that for any $a, b \in I$

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 4 & t > 1. \end{cases}$$

a. Find a formula for $F(x) = \int_0^x f(t)\,dt$.

b. Make a sketch of the graph of F and make a conjecture of where F is continuous.

c. Make a conjecture of where F is differentiable and find $F'(x)$ for all x where you believe F is differentiable.

d. In casual conversation, an analyst might say *integration has a smoothing effect* or *integration increases regularity*. What do they mean by this?