Math 301 — Pointwise and uniform convergence

Problem 1. Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$ for all $n \geq 1$. Find $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$ and prove that $f_n \to f$ pointwise on $[0,1]$. *Note: $f(x)$ is not a continuous function even though $f_n(x)$ is for each $n \geq 1$.*

Problem 2. Let $f_n : [0,\infty) \to \mathbb{R}$ be given by

$$f_n(x) = \frac{nx}{1+nx}$$

for all $n \geq 1$. Find $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0,\infty)$ and prove that $f_n \to f$ pointwise on $[0,\infty)$. *Note: $f(x)$ is not a continuous function even though $f_n(x)$ is for each $n \geq 1$.*

Problem 3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \sin(nx)/n$ for each $n \geq 1$. Find $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in \mathbb{R}$ and prove that $f_n \to f$ uniformly on $\mathbb{R}$.

Problem 4. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(x) = \frac{n}{(n+1)(1+x^2)}$$

for each $n \geq 1$. Find $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in \mathbb{R}$ and prove that $f_n \to f$ uniformly on $\mathbb{R}$.