Math 301 — Properties of uniformly convergent sequences

Problem 1. We have proved a theorem that states:

> If $f_n \to f$ uniformly on D and f_n is continuous on D for all $n \geq 1$, then f is continuous on D.

State the contrapositve of this theorem.

Problem 2. Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$ for all $n \geq 1$. Let $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [0,1]$. Prove that (f_n) does not converge uniformly to f on $[0,1]$.

Problem 3. We have proved a theorem that states:

> If $f_n \to f$ uniformly on $[a,b]$ and f_n and f are integrable on $[a,b]$ for all $n \geq 1$, then $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

State the contrapositve of this theorem.

Problem 4. Let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} nx^n & x \neq 1 \\ 0 & x = 1 \end{cases}$$

for all $n \geq 1$.

a. Find $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [0,1]$.

b. Let $n \geq 1$. Compute $\int_0^1 f_n(x)\,dx$ in terms of n.

c. Prove that (f_n) does not converge uniformly to f on $[0,1]$.