Math 301 — More on uniform convergence

Problem 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$. Use the Weierstrass M-test to prove that f is continuous on $\mathbb{R}$. You may assume without proof that $\cos x$ is continuous on $\mathbb{R}$.

Problem 2. Let $0 < a < 1$. Use the Weierstrass M-test to prove that $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$ to $f(x) = 1/(1-x)$.

Problem 3. In Homework 11 you are asked to prove the following theorem:

> If (f_n) is a sequence of bounded functions on D and $f_n \to f$ uniformly on D then f is bounded on D.

State the contrapositive of this theorem.

Problem 4. Prove that $\sum_{n=0}^{\infty} x^n$ does not converge uniformly to $f(x) = 1/(1-x)$ on $(-1, 1)$.

Problem 5. The following function is known as the *Weierstrass function*. Let a, b be given constants such that $a \in (0, 1)$, b is a positive, odd integer, and $ab > 1+3\pi/2$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x).$$

a. Prove that f is continuous on $\mathbb{R}$.

b. Take a look at the Weierstrass function Wikipedia article to see a depiction of the graph of f. It turns out that f is continuous on $\mathbb{R}$ but nowhere differentiable! Section 38 of our textbook has a longer discussion of such functions that you might find interesting.

Remark. Examples of continuous but nowhere differentiable functions were introduced as early as 1831 by Bernard Bolzano and presented in seminars by Karl Weierstrass in 1872. They did not appear in published work by Weierstrass until 1875 however. Henri Poincaré called them *monsters* and *an outrage against common sense.* Charles Hermite called them a *lamentable scourge* and a *plague.*