Math 301 — Pointwise convergence of functions

Summary. We introduce a simple notion of convergence for sequences of functions (f_n) with a common domain called pointwise convergence. The way it works is that we fix a point x in the domain of f_n and then compute the limit $\lim_{n\to\infty} f_n(x)$ in the way we would for sequences of numbers. For each choice of x, everything is reduced to the theory of Chapter 2. However, we'll see that sometimes the function that we get in the limit isn't necessarily as nice as the individual functions in the sequence.

Problem 1. For each of the following sequences of functions (f_n) with common domain D, try computing the limit

$$\lim_{n\to\infty} f_n(x)$$

for a fixed, but arbitrary, choice of $x \in D$.

a. $f_n(x) = \frac{1}{n(1+x^2)}$, $D = \mathbb{R}$.

b. $f_n(x) = \frac{x^2+nx}{n}$, $D = \mathbb{R}$.

c. $f_n(x) = x^n$, $D = [0,1]$. Careful here: you need to use cases depending on the value of x.

Problem 2. For each sequence in Problem 1, try writing an ϵ-N proof for your limits. That is, given any $\epsilon > 0$ and any $x \in D$, find an N (which might depend on both ϵ and x) such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$, where $f(x)$ is the limit function you found in Problem 1.

a. For which examples does your choice of N depend only on ϵ instead of both ϵ and x? Why would someone call this kind of convergence *uniform convergence*?

b. I've posted some plots of the functions in Problem 1 in the lecture notes pdf online. Try to give an interpretation of the dependence of N on just ϵ or on both ϵ and x in the context of these plots.

Problem 3. If (f_n) is a sequence of continuous functions that converges pointwise, is it always the case that the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ is also continuous? One of the examples in Problem 1 seems to be a counterexample to this. Which one?