Example: Write ε-N proofs for (a), (b), and (c) above. What do you notice about N and its dependence on ε and $x \in D$?

(a) $f_n(x) = \dfrac{1}{n(1+x^2)}$, $f(x) = 0$

Let $\varepsilon > 0$ and let $x \in \mathbb{R}$. Define $N = \underline{1/\varepsilon}$

If $n > N$,

$$|f_n(x) - f(x)| = \left| \frac{1}{n(1+x^2)} - 0 \right|$$

$$= \frac{1}{n(1+x^2)} \qquad 1+x^2 \geq 1$$

$$\leq \frac{1}{n}$$

$$< \frac{1}{N}$$

$$= \varepsilon \qquad \blacksquare$$

(b) $f_n(x) = \dfrac{x^2 + nx}{n}$, $f(x) = x$

Let $\varepsilon > 0$, $x \in \mathbb{R}$. Define $N = \underline{x^2/\varepsilon}$

If $n > N$, then

$$\left| f_n(x) - f(x) \right| = \left| \frac{x^2 + nx}{n} - x \right|$$

$$= \left| \frac{x^2 + nx - nx}{n} \right|$$

$$= \frac{x^2}{n}$$

$$< \frac{x^2}{N} = \varepsilon \quad \blacksquare$$

(c) $f_n(x) = x^n$, $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

Let $\varepsilon > 0$, $x \in [0, 1]$. Define $N = \begin{cases} \log_x \varepsilon & 0 < x < 1 \\ 1 & x = 0, 1 \end{cases}$

Let $n > N$. Suppose $0 < x < 1$. Then

$$|f_n(x) - f(x)| = x^n$$

$$< x^N$$

$$= \varepsilon$$

If $x = 0, 1$

$$|f_n(x) - f(x)| = 0 < \varepsilon \quad \text{for all } n. \quad \blacksquare$$

Question If (f_n) is a sequence of continuous functions on D and $f_n \to f$ pointwise, is f a continuous function on D?

It's not always the case that f is continuous when f_n are all continuous. (c) is a counter-example.