Math 301 — Uniform convergence of functions

Summary. We've now introduced a stronger notion of convergence of sequences of functions. With uniform convergence, we are required to find, given $\epsilon > 0$, a value N such that $n > N$ implies that $|f_n(x) - f(x)| < \epsilon$ for all x in the domain of the functions. The key subtlety in the definition is that the value of N must only depend on ϵ, so that the same N works for all x. We have also learned a computational criterion for determining uniform convergence: $f_n \to f$ uniformly on D if and only if

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0.$$

This worksheet gets us practicing using this criterion.

Problem 1. Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \geq 0$.

a. Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.

b. Let $g_n(x) = f_n(x) - f(x)$. Use the quotient rule to find $g_n'(x)$. Recall that the quotient rule says

$$(f_h/f_l)' = \frac{f_l f_h' - f_h f_l'}{f_l^2}.$$

c. For which values of x is $g_n'(x) = 0$? $g_n'(x) > 0$? $g_n'(x) < 0$?

d. Use part c. to find $\sup\{|g_n(x)| : x \in [0,1]\}$.

e. Use part c. to find $\sup\{|g_n(x)| : x \in [1,\infty)\}$.

f. Does $f_n \to f$ uniformly on $[0,1]$?

g. Does $f_n \to f$ uniformly on $[1,\infty)$?