Math 301 — $\mathbb{Q}$ is dense in $\mathbb{R}$

Summary. We've learned the Archimedean property and its consequences. Now we'd like to use this idea to prove that between any two real numbers, we must be able to find a rational number. This fact is commonly referred to as the fact that the rationals are dense in the reals.

Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$)**.** *For every $a, b \in \mathbb{R}$ such that $a < b$, there exists a rational number $m/n \in \mathbb{Q}$ such that $a < \frac{m}{n} < b$.*

Note 1. Through the rest of the worksheet, we'll assume $a, b > 0$ for simplicity.

Note 2. The idea of our proof will be to show that there is a "mesh" of rational numbers that is fine enough so that there is at least one rational number between a and b. That is, (1) we want a number n so that a multiple of $1/n$ can land between a and b and (2) we want to show that such a multiple m can be found.

Problem 1. Explain why there exists $n \in \mathbb{N}$ such that $1/n$ is smaller than the distance between a and b.

Problem 2. Suppose we have $n \in \mathbb{N}$ such that $1/n$ is smaller than the distance between a and b. Explain why there exists $m \in \mathbb{N}$ such that $m > na$.

Problem 3. Suppose we have $n \in \mathbb{N}$ such that $1/n$ is smaller than the distance between a and b, and suppose that we also have the *smallest* $m \in \mathbb{N}$ such that $m > na$. This means $m - 1 < na$. Why can we conclude that $a < \frac{m}{n} < b$?

Problem 4. Explanations to the previous problems are the core of the proof of the theorem. Let's put everything together in a formal proof. Fill in the blanks below to give a complete proof.

Proof. Our aim is to find $m, n \in \mathbb{N}$ so that $a < m/n < b$. To do this we must first establish that there exists $n \in \mathbb{N}$ so that the increments of size $1/n$ are not too big. By `BLANK1`, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \texttt{BLANK2}. \tag{1}$$

Next, we'll show that we can find a multiple of $1/n$ that fits between a and b. By `BLANK3` there exists $m \in \mathbb{N}$ such that

$$m > na. \tag{2}$$

We choose m to be the smallest such integer, in which case

$$m - 1 < an. \tag{3}$$

Observe that

$$\begin{aligned} a &< \frac{m}{n} \\ &< \frac{an + 1}{n} \\ &= a + \frac{1}{n} \\ &< b, \end{aligned}$$

where the first inequality is due `BLANK4`, the second is due to `BLANK5`, and the third is due to `BLANK6`. Thus, the inequalities above show that $a < \frac{m}{n} < b$. □