Math 339SP, Spring 2024 — Exam 1

Mount Holyoke College

Due March 8 at 5:00 pm

Instructions. This exam consists of 4 questions, with multiple parts, for a total of 50 points. Please do all of them. To receive full credit, you must show your work and provide details and justification where appropriate. You may use your class notes, the textbook, any materials posted to the class web page, and R. However, you may not use other resources (eg. other textbooks or sites on the internet other than our class web page), and I ask that you avoid discussing any aspect of the exam with anyone (except me, Tim). Please submit your work on Gradescope.

Note. I know that you've all been working hard, both in this class and outside of it. I want you to be proud of the effort that you've put in, proud of your individual growth so far, and proud of your integrity. Part of that means taking the honor code seriously, and working on this exam by yourself, without any collaboration or help from classmates or outside materials. I write all of this because I want you to know that I value you each as individuals and value your work and ideas, right or wrong.

Problem 1 (10 points)**.** A bike share program in a certain town has hubs at 7 locations where residents can check out a bike and return it to any hub once finished with their ride. The transition matrix for where bikes are typically returned after one ride, given their check-out location, is given below.

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 3/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 0 & 2/5 & 3/5 \end{bmatrix} \end{array}.$$

1. Find $\lim_{n\to\infty} P^n$ with theory instead of computation. That is, you may use R to check your work, but you must justify your answer with theory.
2. Explain the long term behavior of the system in its real world context. That is, interpret your results in the previous part in a few sentences in terms of the bike share context.
3. Suppose that we follow a bike that starts at hub f. Find the expected number of check-outs until the bike returns to hub f in two distinct ways: (1) using your previous work in this problem and (2) using first-step analysis.

Problem 2 (15 points)**.** A gambler is playing on one of two slot machines, which we call A and B. The gambler wins on machine A with probability $1/6$ and on machine B with probability $1/16$. The gambler will stay on a given machine until they have lost on it twice (total, not in a row). Then they switch machines and continue according to the same rules. For example, their winning and losing sequence might be:

round n	0	1	2	3	4	5	6	7	8	9	10	...
result (win or loss)	W	L	L	W	L	W	L	W	L	W	L	...

For this sequence, they will switch machines after their play at time $n = 2$ since they lost twice. They switch back after $n = 6$ and then again after $n = 10$. The sequence of machines they played on, from rounds 0 to 11, assuming they start on machine A would be $A, A, A, B, B, B, B, A, A, A, A, B, \ldots$.

We can construct a Markov chain from the process described above with state space $\{A_0, A_1, B_0, B_1\}$, where, for example, $X_n = A_0$ indicates round n will be played on machine A and there have been 0 losses during the current turn on machine A. Likewise, $X_n = A_1$ indicates round n will be played on machine A and there has been 1 loss on the current turn on machine A. Similar descriptions hold for states B_0 and B_1.

1. Find the transition matrix P for this Markov chain.
2. Find the probability that the gambler will be playing machine A at time $n = 4$ assuming it's equally likely they start on machine A or B. You may use R to do any matrix computation.
3. Explain why this Markov chain is ergodic.
4. Find the unique stationary distribution of this Markov chain.
5. Find the long-term proportion of rounds played on machine B.

Problem 3 (15 points)**.** For each of the following, either give an example of a Markov chain with the desired properties or explain why it's not possible to give an example. When an example can be given, give a short justification for its validity. You may exhibit the chain in any way that makes the example clear (eg. giving the transition matrix, giving the transition state diagram with edges labeled, declaring it to be a random walk on a certain graph).

1. A Markov chain which has a limiting matrix but not a limiting distribution.
2. A Markov chain which has a limiting distribution but not a limiting matrix.
3. A Markov chain which is aperiodic but does not have a limiting matrix.
4. A Markov chain which is not irreducible but has a limiting distribution.
5. A Markov chain which is irreducible but not ergodic.

Problem 4 (10 points)**.** Let $X_0, X_1, X_2, \ldots$ be a Markov chain with finite state space $\mathcal{S}$ and transition matrix P. Among the following 9 statements, identify the 3 pairs of equivalent statements. This means 6 of the statements will be grouped into distinct pairs and 3 of the statements will go unused. Note that statements A and B are equivalent if $A \Rightarrow B$ and $B \Rightarrow A$ are both true implications. Fill in the table below for your final answer.

1. $P_{ii} > 0$ for all $i \in \mathcal{S}$.
2. $P_{ij} > 0$ for all $i, j \in \mathcal{S}$.
3. $P^n_{ij} > 0$ for all $i, j \in \mathcal{S}$ and all $n \geq 1$.
4. For each $i, j \in \mathcal{S}$ there exists $n \geq 1$ such that $P^n_{ij} > 0$.
5. There exists $n \geq 1$ such that for all $i, j \in \mathcal{S}$, $P^n_{ij} > 0$.
6. There exists a probability distribution π such that $\pi P = \pi$.
7. There exists an initial distribution α so that the distribution of X_n is α for all $n \geq 1$.
8. $X_0, X_1, X_2, \ldots$ is an irreducible Markov chain.
9. $X_0, X_1, X_2, \ldots$ is an aperiodic Markov chain.

Group	Statements
Equivalent pair 1	
Equivalent pair 2	
Equivalent pair 3	
Unused statements	