Math 339SP, Spring 2024 — Exam 2

Mount Holyoke College

Due May 7 at noon

Instructions. This exam consists of 4 questions, with multiple parts, for a total of 50 points. Please do all of them. To receive full credit, you must show your work and provide details and justification where appropriate. You may use your class notes, the textbook, any materials posted to the class web page, and R. However, you may not use other resources (eg. other textbooks or sites on the internet other than our class web page), and I ask that you avoid discussing any aspect of the exam with anyone (except me, Tim). Please submit your work on Gradescope.

Note. I know that you've all been working hard, both in this class and outside of it. I want you to be proud of the effort that you've put in, proud of your individual growth so far, and proud of your integrity. Part of that means taking the honor code seriously, and working on this exam by yourself, without any collaboration or help from classmates or outside materials. I write all of this because I want you to know that I value you each as individuals and value your work and ideas, right or wrong.

Problem 1 (15 points)**.** Arrivals of blue line trains at the Forest Park station form a Poisson process with rate 4 trains per hour, and arrivals of the red line trains form an independent Poisson process with rate 6 trains per hour. The chance that an arriving train has working air conditioning is $1/2$, independently of other trains. Establish a clear random variable notation for the following events, express each event using your notation, and compute its probability, giving a decimal approximation with four significant digits as your answer.

1. The event that at least 3 trains arrive in 1 hour.
2. The event that that it takes longer than an hour for 4 blue lines to arrive.
3. Given that 5 trains arrive in 1 hour, the event that 2 are blue line trains.
4. There is a 45 minute period where no train with working air conditioning arrives.
5. In the time period between two successive blue line train arrivals, no red line trains arrive.

Problem 2 (15 points)**.** Answer the following short answer questions. Give a short justification or counter-example for the true-false questions. No justification is needed for the last two questions.

1. True or false: given only the hold-time parameters q_i and transition matrix $\tilde{P}$ of the embedded discrete-time chain, it is possible to construct the transition function $P(t)$ of a finite-state continuous-time Markov chain.
2. True or false: given the transition function $P(t)$ of a finite-state continuous-time Markov chain, it is possible to find the hold-time parameters q_i and transition matrix $\tilde{P}$ of the embedded discrete-time chain.
3. True or false: given a Poisson process $(N_t)_{t\geq 0}$ the arrival times $S_1, S_2, S_3, \ldots$ are independent, Gamma distributed random variables.
4. Let $(Y_n)_{n\geq 0}$ be a finite-state, discrete-time Markov chain with transition matrix P and let $(N_t)_{t\geq 0}$ be an independent Poisson process with mean rate λ. Define a continuous-time Markov chain $(X_t)_{t\geq 0}$ by $X_t = Y_{N_t}$ for all $t \geq 0$. Find the infinitesimal generator Q of $(X_t)_{t\geq 0}$.

5. Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain with state space $\{1,2,3\}$ and transition function

$$P(t) = \frac{1}{18}\begin{bmatrix} 5+5e^{-6t}+8e^{-3t} & 2-10e^{-6t}+8e^{-3t} & 11+5e^{-6t}-16e^{-3t} \\ 5-7e^{-6t}+2e^{-3t} & 2+14e^{-6t}+2e^{-3t} & 11-7e^{-6t}-4e^{-3t} \\ 5-e^{-6t}-4e^{-3t} & 2+2e^{-6t}-4e^{-3t} & 11-e^{-6t}+8e^{-3t} \end{bmatrix}.$$

   Find all stationary distributions of $(X_t)_{t\geq 0}$.

Problem 3 (15 points)**.** A single repair worker is available to service two machines. The first machine breaks down on average 3 times per hour. The second machine breaks down on average once per hour. It takes on average 30 minutes to repair the first machine. The first machine has higher priority, so the repair worker will always stop what they're doing and service it when it breaks. That is, even if they're in the middle of working on machine 2, the repair worker will stop and go to service machine 1 until it is fixed, losing any progress made on repairing machine 2. Machine 2 takes on average 15 minutes of uninterrupted time to repair. Assume all service and breakdown times are exponentially distributed and independent.

1. Describe a state space to model this system and draw a transition state diagram.
2. Find the stationary distribution of the system.
3. Find the long term proportion of time that machine 2 is broken.
4. Find the long term expected value of the number of broken machines.

Problem 4 (5 points)**.** Write a short reflection on your experience in the class. Did you have a favorite topic? A favorite idea or theorem? What do you see as the overarching story of the class? Do you see connections to other things you have studied in math, stat, computer science, or some other field? Feel free to respond to any of these guiding questions. Your answer should be about a page long.