# Math 339SP, Spring 2022 — Homework 2

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Due February 11 at 5:00 pm

**Instructions.** This problem set covers material from Week 2 of class, with a focus on Chapter 2 of the textbook.

**Problem 1.** Watch the following Numberphile video that features the probabilist Persi Diaconis.

1. If one views the process of shuffling a deck of cards as a Markov chain, what are the states of the Markov chain and how big is the state space?
2. The video discusses a theorem that says it takes 7 *riffle* shuffles for a deck of cards to be well mixed. If $P$ is the transition matrix of the Markov chain you described above, how can this theorem be summarized in terms of $P$? What is the corresponding theorem for the *overhand* shuffle?
3. What are you most curious about after watching the video?

**Problem 2.** Try the following exercises from Chapter 2.

1. Exercise 2.3 (make sure to read about the Wright-Fisher model in Example 2.6 first)
2. Exercise 2.11
3. Exercise 2.16 (I want you to do a special case of this problem and just show that $P^2 = P$. The general case of $P^n = P$ can be proved by induction and you can try this for fun but it's not required. Note that a stochastic matrix is one with the property that, for any given row, its entries sum to 1.)
4. Exercise 2.18 (Note that you're assuming that the initial distribution is $\alpha = (1/k, 1/k, \ldots, 1/k)$ and you're trying to prove that $P(X_n = j) = 1/k$ for any $j = 1, \ldots, k$.)
5. Exercise 2.26

**Problem 3.** Consider a 3-state Markov chain which is constructed as follows. Let

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}, \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

be $3 \times 3$ stochastic matrices (which correspond to two other Markov chains with the same state space). Before each step of our Markov chain, we toss a fair coin. If it lands heads, we will transition to a new state according to the probabilities in $Q$; if it lands tails, we will use $R$.

1. Give the transition matrix $P$ of our Markov chain, entry by entry, in terms of the entries of $Q$ and $R$.
2. Express $P$ in terms of $Q$ and $R$ without writing out the entries. For example, if you think $P$ is just the sum of $Q$ and $R$, then you'll write $P = Q + R$.
3. Modify your previous answer for the following case: when the coin lands tails, instead of transitioning according to the probabilities in $R$, we stay in our current state with probability 1.

**Problem 4.** Do Problem 2 from the worksheet on Thursday, February 3.