Math 339SP, Spring 2022 — Homework 3

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Due February 18 at 5:00 pm

Instructions. This problem set covers material from Week 3 of class, with a focus on Chapter 3 of the textbook.

Problem 1. Suppose a 3-state Markov chain has transition matrix given by

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.$$

1. Solve a system of equations to find a probability vector π such that $\pi P = \pi$. You may use R to do row reduction.
2. Give $\lim_{n\to\infty} P^n$ without using technology. Cite a result from class to justify your answer.
3. Read the end of Section 3.1 of our textbook starting at the heading *Proportion of Time in Each State*. Give the expected long run proportion of visits to each of the 3 states.

Problem 2. Recall the random walk on the 6-cycle that was discussed in Week 3 worksheets. We found that it had a unique stationary distribution but that it did not have a limiting distribution. Here, we introduce a modification of this Markov chain called a *lazy random walk*. Before each step of the lazy random walk on the 6-cycle, we toss a coin. If it lands heads, the walker transitions to one of its neighbors like in the standard random walk; if it lands tails, the walker stays at its current vertex.

1. Express the transition matrix Q for the lazy random walk in terms of the transition matrix P for the standard random walk. Feel free to use Problem 3 from Homework 2.
2. Prove that $\pi Q = \pi$, where π is the stationary distribution you found for P.
3. Is Q regular? You may use technology to justify your answer.
4. Find $\lim_{n\to\infty} Q^n$ without using technology. Cite a result from class to justify your answer.
5. Informally, we mentioned that *periodicity* was the obstruction to the random walk on the 6-cycle having a limiting distribution. We will soon define the term periodicity formally, but for now we can think of a state being *n-periodic* when we're only able to return to it in multiples of n steps. What is the periodicity of each state in the standard random walk on the 6-cycle? What about the lazy version? No justification is needed for these questions.

Problem 3. Try the following exercises from Chapter 3.

1. Exercise 3.5 (always feel free to use R to row reduce and solve systems of equations)
2. Exercise 3.7 (I want you to do the special case of $n = 5$ instead of the general case. It's possible to do this problem with a minimal amount of computation. Feel free to do this problem using R to solve a system of equations if you get stuck though.)
3. Exercise 3.8 (If the chain does not have a unique stationary distribution, describe all the stationary distributions.)
4. Exercise 3.10