Math 339SP, Spring 2022 — Homework 5

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Due March 4 at 5:00 pm

Instructions. This problem set covers material from Week 5 of class, with a focus on Chapter 3 of the textbook.

Problem 1. We will soon learn a theorem called the Limit Theorem for Ergodic Markov chains. We define a Markov chain to be *ergodic* if it has finitely many states, is irreducible (ie. one communication class), and is aperiodic (all states have period 1). The theorem states that an ergodic Markov chain has a limiting distribution (whose components are all positive and which is the unique stationary distribution of the Markov chain). The theorem serves as a better version of the Limit Theorem for Regular Matrices since irreducibility and aperiodicity are hypotheses that can be checked without having to resort to investigating powers of the transition matrix, and thus are generally more practical. That is, it is sometimes impractical to write down the transition matrix for a Markov chain. Explain why the following Markov chains are ergodic and thus have a limiting distribution.

1. The lazy random walk on a connected graph with finitely many vertices.
2. A Markov chain with state space $\{1, \ldots, k\}$ that evolves as follows. Before each step, a coin is tossed. If it lands heads, the system stays in its current state. If it lands tails, it moves to a different state with all $k-1$ other states equally likely.
3. A Markov chain on a graph (where each vertex has at least one neighbor, but the graph has possibly disconnected components) that evolves as follows. Before each step, a coin is tossed. If it lands heads, the next state is chosen according to the usual rules of a random walk: the walker moves to one of its neighbor vertices at random. If the coin lands tails, the next state is chosen uniformly at random with all vertices in the graph equally likely.

Problem 2. Try the following exercises from Chapter 3.

1. Exercise 3.11
2. Exercise 3.15 (Read about the formula for the stationary distribution of a random walk on a graph in Example 3.8. Also note that a chess board has 64 squares. Finally, note that the knight is allowed to up and down or left and right to adjacent squares but is not allowed to move diagonally.)
3. Exercise 3.18
4. Exercise 3.19