Math 339SP, Spring 2022 — Homework 7

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Due April 1 at 5:00 pm

Instructions. This problem set covers material from Week 9 of class, with a focus on Chapter 6 of the textbook.

Problem 1. Try the following exercises from Chapter 6.

1. Exercise 6.3
2. Exercise 6.4
3. Exercise 6.5
4. Exercise 6.6. This problem will get reduced to solving an equation of the form $f(\lambda) = 0$ that can only be solved numerically. That is, a closed form expression for λ cannot be given. State the equation that must be solved and then use the R function `uniroot` to give a numerical approximation of its solution. The code below gives an example of its usage. The first line sets up a function f. The second line looks for a solution to the equation $f(x) = 0$ in, for example, the interval $(3, 4)$.

```
f = function(x) sin(x)
uniroot(f, c(3,4))$root
```

5. Exercise 6.13, part a. Your goal in this problem is to take $P(N_s = k \mid N_t = n)$ and simplify it to reveal that the conditional distribution of N_s given $N_t = n$ is binomial. Your solution should give the steps of your derivation and conclude with the parameters of the binomial distribution.)

Problem 2. Our end of semester project is coming up and I'd like us to start thinking about it a little. The goal of the project will be for you to independently learn and communicate about a topic on stochastic processes that hasn't been discussed in class. You'll work in groups of 2 or 3, give a short presentation, and write a summary report. I have some suggested topics listed below. As your submission for this homework, please tell me which topic(s) interest you, giving a short explanation of why, and tell me whether you have preferences for groupmates. Please feel free to state a preference for a topic that's not listed below. This will help me assign groups.

1. Random walks on $\mathbb{Z}^d$. We've studied random walks on graphs in class, but never on graphs with infinitely many vertices. The random walk on $\mathbb{Z}^d$ for an integer $d \geq 1$ is a simple process to describe but has a rich behavior. A project here could involve a discussion of recurrence and transience for random walks on $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3$, and the theorem that the probabilist Shizuo Kakutani summarized as "A drunk man will find his way home, but a drunk bird may get lost forever." See Example 3.13 in the textbook.
2. Markov chain Monte Carlo, the Metropolis Hastings algorithm, and stochastic optimization. Here is a simple, but surprisingly deep question that has driven research in probability theory since the 1950s: given an arbitrary probability distribution π on a state space $\mathcal{S}$, can you simulate a random variable X whose distribution is π (ie. write an algorithm to sample random numbers

distributed as π). This is the question at the core of Monte Carlo algorithms, and Markov Chain Monte Carlo (MCMC) algorithms attempt to address it by constructing ergodic Markov chains whose limiting distribution is the prescribed distribution π. There is a surprisingly effective and beautiful connection between MCMC algorithms and the problem of finding the max/min of a function whose domain is $\mathcal{S}$, which has importance in all kinds of important applications even in the case when $\mathcal{S}$ is a finite set. There are actually a few possible projects here, each describing an optimization problem, and how it can be solved using something called the Metropolis Hastings algorithm.

(a) MCMC and the traveling salesman problem; See Section 5.2 and Wikipedia for a description of the traveling salesman problem

(b) MCMC and the knapsack problem See Section 5.2 and Wikipedia for a description of the knapsack problem

(c) MCMC and cracking encryptions; Section 5.2, Example 5.3, and the introduction of the paper *The Markov Chain Monte Carlo Revolution* by Persi Diaconis

3. Branching processes. Consider the following model of a population that evolves in discrete time steps (generations). In a given generation, each individual gives rise to a random (i.i.d.) number of offspring that will form the population of the next generation. Such a process, with origins in the study of genealogy, has applications in biology, epidemeology, spread of computer viruses, among other areas. The main question in a first study of branching processes is the following: under what conditions on the offspring distribution does the population go extinct or, conversely, grow without bound? See Chapter 4.

4. Brownian motion. All of our stochastic processes of study have had discrete state space, but there is a rich class of continuous state space, continuous time processes. The most fundamental is called Brownian motion, which has applications in all kinds of areas like physics and mathematical finance. A project here could be an introduction to Brownian motion, its basic properties, how it's simulated, and its relationship to the random walk on $\mathbb{Z}$. See Sections 8.1 and 8.2.