

Math 339SP, Spring 2024 — Homework 9

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Due April 12 at 5:00 pm

Instructions. This problem set covers material from Week 11 of class, with a focus on Chapter 7 of the textbook.

Problem 1. During lunch hour, customers arrive at a fast-food restaurant at a rate of 120 customers per hour. The restaurant has one line, with three workers taking orders at independent service stations. Each worker takes an exponentially distributed amount of time—on average 1 minute—to service a customer. Let X_t denote the number of customers in the restaurant (in line and being served) at time t .

1. Give the state space of this Markov chain.
2. Draw the transition rate diagram and label transition rates.
3. If there are 2 customers in the restaurant currently, find the probability a new customer will arrive before one of the 2 customers in the restaurant leaves.
4. If there are 5 customers in the restaurant currently, find the probability a new customer will arrive before one of the 5 customers in the restaurant leaves.

Problem 2. For the fast-food restaurant of the previous exercise, assume that customers turn away from the store if all three service stations are busy. Let Y_t denote the number of service stations busy at time t .

1. Give the state space of this Markov chain.
2. Draw the transition rate diagram and label transition rates.
3. Give the hold time distribution for each state.
4. Give the embedded chain transition matrix \tilde{P} .

Problem 3. Consider the Markov chain with transition rate diagram given in Figure 7.12 on page 315.

1. Give the hold time distribution for each state.
2. Give the embedded chain transition matrix \tilde{P} .
3. Give the infinitesimal generator Q .
4. Suppose $X_0 = 1$. Find the probability of
 - (a) being in state 1 at time 0.75
 - (b) being in state 2 at time 1.2
 - (c) being in state 3 at time 2.3

Problem 4. In class we learned about the $M/M/1$ queue model, where arrivals occur according to a Poisson process with rate λ , service times are independent and exponentially distributed with parameter μ , there is 1 server, and X_t denotes the number of customers in line or being serviced at time t . Draw the transition rate diagram and label transition rates for the $M/M/c$ queue where arrivals and service times are modeled at the same rates, but there are $c \in \mathbb{N}$ servers that give service to customers in parallel and independently.

Problem 5. Try the following exercises from Chapter 7.

1. Exercise 7.22
2. Exercise 7.23a, 7.23c