- Tim / Prof. / Prof. Chumley (he/him)
- Moodle - announcements
- Webpage (tchumley.mtholyoke.edu/m339sp)
  - notes, worksheets, homework, syllabus
  - updated daily
- Homework (weekly)
  - submitted on Gradescope
  - due Fridays at 5 pm
- Exams - one during semester, one during finals
- Project - second half of semester, learn a new topic or read a research paper and present
- Participation - come to class, be a good community member, stay in touch when something goes wrong (eg. illness)
- Office hours (tentative)
  - Mondays 4:00 - 5:00
  - Wednesdays 1:00 - 2:00
  - Fridays 11:00 - 12:00

  drop in (Clapp 423), no appointment necessary

this week 11:00 - 12:00

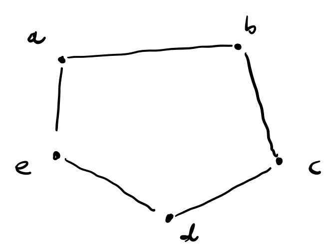
Chapter 1 Introduction to Markov chains

Def A stochastic process is an ordered set of random variables $\{X_t : t \in I\}$ which all take on values in a set S called the state space of the process. The set I is called the index set and it represents time. Typical index sets are

$$I = \{0, 1, 2, 3, \dots\} \quad (\text{discrete time})$$

and $$I = [0, \infty) \quad (\text{continuous time}).$$

Example (board game) Consider the following undirected graph with vertices and edges:



The vertices might represent spaces on a board game like Monopoly. Suppose we roll a die and move clockwise along the board from our current vertex according to the value of the roll.

Then $S = \{a, b, c, d, e\}$ (location of the player)

$I = \{0, 1, 2, 3, \dots\}$ (rounds of the game)

and $\{X_0, X_1, X_2, \dots\}$ is a stochastic process

where X_n represents the position on the board

after the n^{th} roll.

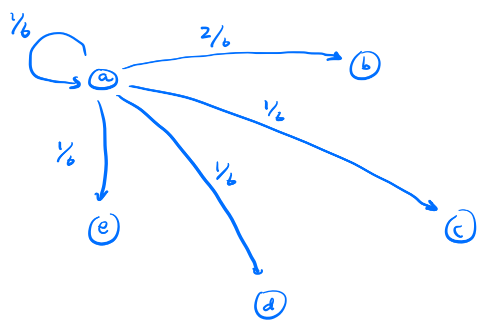
Def (informal for now) A Markov chain is a stochastic process with the following property:

the distribution of future states depends only on the present state, not on past states.

Remark our board game example is a Markov chain.

Example (board game) The transition state diagram of a Markov chain is a directed graph where the vertices are the states in S and edges are labeled with one-step transition probabilities between states.

Let's draw part of the transition state diagram for the board game.



This diagram is not yet complete. It only captures transitions starting from state a. Try to complete it yourself.

Notice the edge probabilities eminating from each state should sum to 1.

The <u>transition matrix</u> of a Markov chain encodes transition probabilities from each state.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} &
\begin{bmatrix}
1/6 & 2/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 2/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 2/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 2/6 \\
2/6 & 1/6 & 1/6 & 1/6 & 1/6
\end{bmatrix}
\end{array}
$$

Row a represents transition probabilities away from state a.

Notice the rows each sum to 1.

Problem 1. For each of the following examples, identify a stochastic process $\{X_t : t \in I\}$. Then, determine the state space S and the index set I.

a. We're interested in modeling day to day weather; specifically, we're interested in rain conditions. Suppose that the chance of rain tomorrow depends only on today's conditions and not on any past conditions. If it rains today, then it will rain tomorrow with probability 0.6. If it does not rain today, then it will rain tomorrow with probability 0.2.

b. A gambler starts with \$5 and decides to play the following game. In each round, a coin is tossed. If the coin lands on heads, the gambler wins \$1; otherwise the gambler loses \$1. The gambler stops playing when they run out of money or when they have \$10 (that is, \$5 more than what they started with).

c. A small grocery store has one employee whose job is to work the cashier. When a customer arrives at the cashier line, if no one is in line, they are served. Any other customers that arrive at the line while the employee is busy must wait to be served. However, if the line is too long (ie. there are 10 or more people), they leave. What if there's no limit to the length of the line?

(a) $S = \{R, N\}$ states representing rain and non-rain conditions

$I = \{0, 1, 2, 3, \dots\}$ representing discrete days

(b) $S = \{0, 1, 2, \dots, 10\}$ representing amount of money gambler has

$I = \{0, 1, 2, 3, \dots\}$ representing rounds of the game

(c) $S = \{0, 1, 2, 3, \dots, 10\}$ representing number of customers in line or being served, or $S = \{0, 1, 2, 3, \dots\}$ when there is no limit to the length of the line.

$I = [0, \infty)$ representing continuous time since the store's opening.

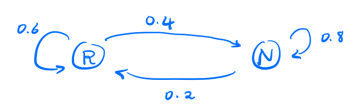
Problem 2. Let's focus on the weather example above. Part c below might feel a little tricky but we'll review probability ideas together all semester and we'll learn new ideas related to matrix algebra that will make these and trickier questions more routine.

a. Draw a transition state diagram and write the transition matrix for this example.

b. Suppose today is Day 0 and it's raining. What is the probability that it's raining on Day 1? There's no computation to do here; I just want you express what we're looking for using random variable and conditional probability notation.

c. Suppose today is Day 0 and there's 50% chance it's raining. What is the probability that it's raining on Day 1? Before giving a numerical answer, express what we're looking for using random variable and conditional probability notation.

(a)

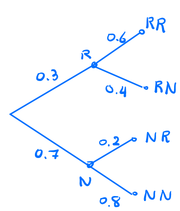


$$\begin{array}{cc} & \begin{array}{cc} R & N \end{array} \\ \begin{array}{c} R \\ N \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

(b) $P(X_1 = R \mid X_0 = R) = 0.6$

(c) We're given $P(X_0 = R)$ and we want to find $P(X_1 = R)$.

Here is a tree diagram showing 2-day sequences of weather



$P(X_1 = R)$

$= P(X_1 = R, X_0 = R)$

$\quad + P(X_1 = R, X_0 = N)$

$= (0.3)(0.6) + (0.7)(0.2)$

$= 0.18 + 0.14 = 0.32$

Next time we'll talk about the Law of Total Probability.