§ 3.4 Irreducible Markov Chains

Def Let $T_i = \min\{n > 0 : X_n = i\}$. This is called the first hitting time for state i. The expected first return time to state i is $E[T_i \mid X_0 = i]$.

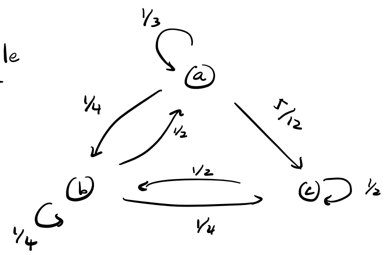
Theorem A finite state, irreducible Markov chain has a unique stationary distribution π where

$$\pi_i = \frac{1}{E[T_i \mid X_0 = i]}$$ for each state i.

Intuition Since all states are recurrent, each state is visited infinitely often and $\frac{1}{E[T_i \mid X_0 = i]}$ is keeping track of the frequency of visits to i. Note the chain might not have a limiting distribution, but this theorem guarantees it has a unique stationary distribution.

Computational idea: we can solve for the stationary distribution using linear algebra (RREF) to get information of first return times, but the converse is possible as well: we compute expected first times directly to get the stationary distribution.

Example



Let's compute $E[T_a \mid X_0 = a]$ using a technique we'll call first step analysis.

Let $e_x = E[T_a \mid X_0 = x]$ for $x = a, b, c$.

Then

$$
\begin{aligned}
e_a &= E[T_a \mid X_0 = a] \\
&= \tfrac{1}{3} E[T_a \mid X_0 = a, X_1 = a] + \tfrac{1}{4} E[T_a \mid X_0 = a, X_1 = b] \\
&\qquad + \tfrac{5}{12} E[T_a \mid X_0 = a, X_1 = c] \\
&= \tfrac{1}{3} + \tfrac{1}{4}\left(1 + E[T_a \mid X_0 = b]\right) + \tfrac{5}{12}\left(1 + E[T_a \mid X_0 = c]\right) \\
&= \tfrac{1}{3} + \tfrac{1}{4}(1 + e_b) + \tfrac{5}{12}(1 + e_c)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
e_b &= E[T_a \mid X_0 = b] \\
&= \tfrac{1}{2} E[T_a \mid X_0 = b, X_1 = a] + \tfrac{1}{4} E[T_a \mid X_0 = b, X_1 = b] \\
&\qquad + \tfrac{1}{4} E[T_a \mid X_0 = b, X_1 = c] \\
&= \tfrac{1}{2} + \tfrac{1}{4}(1 + e_b) + \tfrac{1}{4}(1 + e_c)
\end{aligned}
$$

and

$$
\begin{aligned}
e_c &= E[T_a \mid X_0 = c] \\
&= (0) E[T_a \mid X_0 = c, X_1 = a] + \tfrac{1}{2} E[T_a \mid X_0 = c, X_1 = b]
\end{aligned}
$$

$$+ \ \tfrac{1}{2} E[T_a \mid \overline{X}_0 = c, \overline{X}_1 = c]$$

$$= \tfrac{1}{2}(1+e_b) + \tfrac{1}{2}(1+e_c)$$

This gives a system of 3 equations and 3 unknowns:

$$\begin{cases} e_a = \frac{1}{3} + \frac{1}{4}(1+e_b) + \frac{5}{12}(1+e_c) \\ e_b = \frac{1}{2} + \frac{1}{4}(1+e_b) + \frac{1}{4}(1+e_c) \\ e_c = \frac{1}{2}(1+e_b) + \frac{1}{2}(1+e_c) \end{cases}$$

$$\Longrightarrow \begin{cases} e_a - \frac{1}{4}e_b - \frac{5}{12}e_c = 1 \\ \frac{3}{4}e_b - \frac{1}{4}e_c = 1 \\ -\frac{1}{2}e_b + \frac{1}{2}e_c = 1 \end{cases}$$

Solve this using R and the RREF function.

$$\left[\begin{array}{ccc|c} 1 & -1/4 & -5/12 & 1 \\ 0 & 3/4 & -1/4 & 1 \\ 0 & -1/2 & 1/2 & 1 \end{array}\right]$$

```
# Finding E[T_a | X_0 = a]

```{r}
A = matrix(c(1, -1/4, -5/12, 1,
             0, 3/4, -1/4, 1,
             0, -1/2, 1/2, 1), nrow = 3, ncol = 4, byrow = T) # an augmented matrix
fractions(rref(A)) # the command to find its reduced row echelon form
```

     [,1] [,2] [,3] [,4]
[1,]   1   0   0 23/6
[2,]   0   1   0   3
[3,]   0   0   1   5
```

So $e_a = 23/6$

$e_b = 3$

$e_c = 5$

This tells the expected first return time to a is $\frac{23}{6}$ and $\pi_a = \frac{6}{23}$.

Warning this doesn't tell us about π_b nor π_c.

Exercise Set up similar systems for finding

$E[T_b \mid X_0 = b]$ and $E[T_c \mid X_0 = c]$.

$E[T_b \mid X_0 = b]$

Let $e_x = E[T_b \mid X_0 = x]$, $x = a, b, c$.

$$\begin{cases} e_a = \frac{1}{3}(1+e_c) + \frac{1}{4} + \frac{5}{12}(1+e_c) \\ e_b = \frac{1}{2}(1+e_a) + \frac{1}{4} + \frac{1}{4}(1+e_c) \\ e_c = \frac{1}{2} + \frac{1}{2}(1+e_c) \end{cases}$$

$E[T_c \mid X_0 = c]$.

Let $e_x = E[T_c \mid X_0 = x]$, $x = a, b, c$

$$\begin{cases} e_a = \frac{1}{3}(1+e_a) + \frac{1}{4}(1+e_b) + \frac{5}{12} \\ e_b = \frac{1}{2}(1+e_a) + \frac{1}{4}(1+e_b) + \frac{1}{4} \\ e_c = \frac{1}{2}(1+e_b) + \frac{1}{2} \end{cases}$$