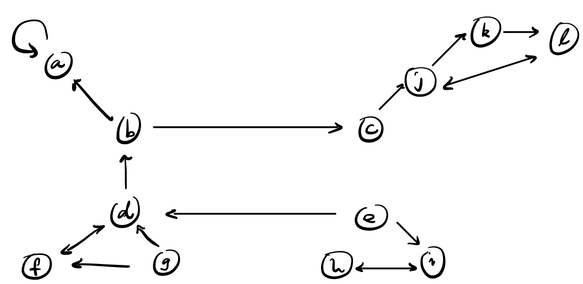
§ 3.5, 3.6 Periodicity and Ergodicity

Def The *period* of a state i, denoted $d(i)$, is given by $d(i) = \gcd\{n > 0 : P_{ii}^{n} > 0\}$. In words, it's the greatest divisor of the set of possible return times to i. If $d(i) = 1$, state i is called *aperiodic*. If the set of possible return times of i is empty, we declare $d(i) = +\infty$.

Example Find communication classes and periods of states for the Markov chain with transition state diagram:



Comm. classes: $\{a\}, \{b,c,d,e,f\}, \{g\}, \{h,i\}, \{j,k,l\}$

Periods: $d(a) = 1$, $d(b) = 2$, $d(g) = +\infty$, $d(h) = 2$, $d(j) = 1$

Lemma The states of a comm. class all have the same period. In other words, periodicity is a class property.

Proof Let $i \sim j$ be in the same communication class.

We'll try to show $d(i) = d(j)$ by showing $d(i) \leq d(j)$ and $d(j) \leq d(i)$. We'll show that $d(i) \leq d(j)$ by showing that $d(i)$ is a divisor of the set $\{n > 0 : P_{jj}^n > 0\}$ and than exploit the fact that $d(j)$ is the greatest divisor of this set. To show that $d(i)$ is a divisor of this set, we must that for any $n > 0$ such that $P_{jj}^n > 0$, there exists $m \in \mathbb{Z}$ such that $d(i) m = n$.

Let $n > 0$ such that $P_{jj}^n > 0$. Since $i \sim j$, there exist $r, s \geq 0$ such that $P_{ij}^r > 0$ and $P_{ji}^s > 0$.

We claim that $r+s$ and $r+s+n$ are possible return times to i. Indeed, notice that

$$
\begin{aligned}
P_{ii}^{r+s} &= P(X_{r+s} = i \mid X_0 = i) \\
&\geq P(X_{r+s} = i, X_r = j \mid X_0 = i) \\
&= P(X_{r+s} = i \mid X_r = j, X_0 = i) P(X_r = j \mid X_0 = i) \\
&= P_{ji}^s P_{ij}^r > 0
\end{aligned}
$$

and

$$
\begin{aligned}
P_{ii}^{r+s+n} &= P(X_{r+s+n} = i \mid X_0 = i) \\
&\geq P(X_{r+s+n} = i, X_{r+n} = j, X_r = j \mid X_0 = i) \\
&= P_{ij}^r \cdot P_{jj}^n P_{ji}^s > 0.
\end{aligned}
$$

Therefore $d(i)$ divides $r+s$ and $r+s+n$. That is, there exist $k, \ell \in \mathbb{Z}$ such that $d(i)k = r+s$ and $d(i)\ell = r+s+n$. This implies

$$d(i)k - d(i)\ell = n \implies d(i)(k-\ell) = n$$

which implies $d(i)$ divides n. Therefore $d(i) \leq d(j)$. The same argument with the roles of i and j reversed shows that $d(j) \leq d(i)$.

<u>Def</u> A Markov chain is called <u>aperiodic</u> if all its states are aperiodic. A Markov chain is called <u>ergodic</u> if it is (1) irreducible, (2) aperiodic, and (3) all states have finite expected return times (which is true for all irreducible chains with finite state space).

<u>Theorem</u> (Limit Theorem for Ergodic Markov Chains)

An ergodic Markov chain always has a limiting distribution. In particular it has a unique stationary distribution, with strictly positive components, that is also a limiting distribution.

Idea irreducibility insures all states communicate (i.e. canonical form of P has a single block, so no issues of different rows in limiting matrix, aperiodicity means $\lim_{n\to\infty} P^n$ really exists so that there are no issues like P^{2n} being different from P^{2n+1} in random walk on 6-cycle)

Theorem (Limit Theorem for Regular Markov chains)
A Markov chain with a regular transition matrix always has a limiting distribution. In particular it has a unique stationary distribution, with strictly positive components, that is also a limiting distribution.

Idea This is an equivalent theorem with a linear algebraic hypothesis instead of a probabilistic hypothesis. The regular matrix requirement ends up allowing you to do something like diagonalize P (specifically it has a Jordan canonical form $\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & Q \\ 0 & \end{pmatrix}$ where $Q^n \to 0$ as $n\to\infty$), $\lambda = 1$ is an eigenvalue with a unique corresponding probability eigenvector and all other eigenvalues strictly less than 1.