3.8 Absorbing chains

Def State i is called absorbing if $P_{ii} = 1$.

If a chain has at least one absorbing state, it's called an absorbing chain.

Goal Understand behavior of P^n for absorbing chains.

Suppose a chain has t transient states and $k-t$ absorbing states.

Reordering states as necessary, P has the form.

$$P = \begin{array}{c} \\ \text{transient} \\ \text{absorbing} \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} \text{transient} & \text{absorbing} \end{array} \\ \left(\begin{array}{c|c} Q & R \\ \hline O & I \end{array}\right) \\ \begin{array}{cc} \underbrace{\qquad}_{t} & \underbrace{\qquad}_{k-t} \end{array} \end{array} \begin{array}{l} \\ \}\, t \\ \}\, k-t \\ \\ \end{array}$$

Then $$P^2 = \left(\begin{array}{c|c} Q & R \\ \hline O & I \end{array}\right)\left(\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} Q^2 & QR+R \\ \hline 0 & I \end{array}\right),$$

$$P^3 = \left(\begin{array}{c|c} Q^2 & QR+R \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} Q^3 & Q^2R+QR+R \\ \hline 0 & I \end{array}\right),$$

and $$P^n = \left(\begin{array}{c|c} Q^n & (I+Q+\cdots+Q^{n-1})R \\ \hline 0 & I \end{array}\right)$$

Questions 1) What is $\lim_{n\to\infty} Q^n$?

$=$ $t\times t$ O-matrix since the long-term probability of being in a transient state is 0.

2) What is $\lim_{n\to\infty} (I+Q+\cdots+Q^{n-1})R$?

what does it represent probabilistically?

it represents the long-term probability of getting absorbed in a particular state.

Lemma $\displaystyle \lim_{n\to\infty} (I + Q + \cdots + Q^{n-1}) = \sum_{k=0}^{\infty} Q^k = (I-Q)^{-1}$

as long as $Q^k \to 0$ as $k \to \infty$.

Proof Let $S_n = I + Q + \cdots + Q^n$. Then

$$(I-Q)S_n = S_n - QS_n$$

$$= (I + Q + \cdots Q^n) - (Q + Q^2 + \cdots + Q^{n+1})$$

$$= I - Q^{n+1}$$

Taking the limit as $n \to \infty$ of both sides,

$$(I-Q)\sum_{n=0}^{\infty} Q^n = I$$

As long as $I-Q$ is invertible, $\displaystyle\sum_{n=0}^{\infty} Q^n = (I-Q)^{-1}$

Claim $I-Q$ is invertible.

This is equivalent to $(I-Q)x = 0$ having a unique solution $(x=0)$. This is the case because if x is any solution then $(I-Q)x = 0 \Rightarrow x = Qx$

$$\Rightarrow x = Q^2x \Rightarrow \cdots \Rightarrow x = Q^n x \text{ for all } n$$

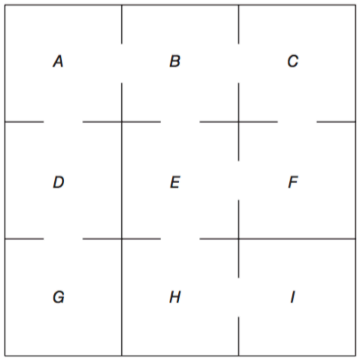
$$\Rightarrow x = \lim_{n\to\infty} Q^n x = 0$$

Conclusion

$$\lim_{n\to\infty} P^n = \begin{array}{c|c} & \begin{array}{cc} t & a \end{array} \\ \hline \begin{array}{c} t \\ a \end{array} & \left(\begin{array}{c|c} 0 & (I-Q)^{-1}R \\ \hline 0 & I \end{array}\right) \end{array}$$

and $[(I-Q)^{-1}R]_{ij}$ represents the long-term probability of starting in transient state i and being absorbed in absorbing state j.

Problem 1. A mouse is placed in the maze below, starting in room A. A (humane) trap is placed in room F and a piece of cheese is placed in room I. From each room, the mouse moves to an adjacent room through an open door, choosing from available doors with equal probability. What is the probability the mouse finds the cheese before the trap? Note that the R command `solve(eye(7)-Q)` can be used to compute $(I - Q)^{-1}$ when I and Q are 7×7 matrices.



```{r}
Q = matrix(c(0, 1/2, 0, 1/2, 0, 0, 0,
             1/3, 0, 1/3, 0, 1/3, 0, 0,
             0, 1/2, 0, 0, 0, 0, 0,
             1/2, 0, 0, 0, 0, 1/2, 0,
             0, 1/3, 0, 0, 0, 0, 1/3,
             0, 0, 0, 1, 0, 0, 0,
             0, 0, 0, 0, 1/2, 0, 0), nrow = 7, ncol = 7, byrow = T)
colnames(Q) =  c("A", "B", "C", "D", "E", "G", "H")
rownames(Q) = colnames(Q)
R = matrix(c(0, 0,
             0, 0,
             1/2, 0,
             0, 0,
             1/3, 0,
             0, 0,
             0, 1/2), nrow = 7, ncol = 2, byrow = T)
rownames(R) = colnames(Q)
colnames(R) = c("F", "I")
# solve(eye(7) - Q) %*% R # full matrix of absorption probabilities
(solve(eye(7) - Q) %*% R)["A", "I"] # probability of absorption at I, given starting at A
```

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[1] 0.1818182
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