Theorem Let i and j be transient states and let F_{ij} be the expected number of visits to j, given $X_0 = i$. Then $F_{ij} = [(I-Q)^{-1}]_{ij}$

Proof Let $I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{if } X_n \neq j. \end{cases}$ Then

$$F_{ij} = E\left[\sum_{n=0}^{\infty} I_n \,\middle|\, X_0 = i \right]$$

$$= \sum_{n=0}^{\infty} E[I_n \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} P^n_{ij} = \sum_{n=0}^{\infty} Q^n_{ij} = [(I-Q)^{-1}]_{ij}$$

Corollary Suppose $\{1,\dots,t\}$ are transient states and $\{t+1,\dots,k\}$ are absorbing states in a k-state chain. Suppose i is a transient state and $X_0 = i$. Then the expected number of steps before absorption is $F_{i1} + F_{i2} + \cdots F_{it}$.

Summary Let i and j be transient states and a and absorbing state. Suppose $X_0 = i$.

1) Prob. of absorption at a is $[(I-Q)^{-1}R]_{ia} = [FR]_{ic}$

2) expected time spent at j before absorption is

$$[(I-Q)^{-1}]_{ij} = F_{ij}$$

3) expected time spent before absorption is

$$[(I-Q)^{-1}]_{i1} + \cdots + [(I-Q)^{-1}]_{it}$$

$$= F_{i1} + \cdots + F_{it}$$

(assuming states $1,\dots,t$ are the transient ones)

Example We are given a biased coin with heads prob. $2/3$. Find the expected number of flips needed to get 3 H in a row. Define a Markov chain (X_n) with state space $\{\phi, H, HH, HHH\}$ and transition matrix

$$P = \begin{array}{c|c} & \begin{array}{cccc} \phi & H & HH & HHH \end{array} \\ \begin{array}{c} \phi \\ H \\ HH \\ HHH \end{array} & \left(\begin{array}{ccc|c} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 1/3 & 0 & 0 & 2/3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

The states of our Markov chain represent the most recent run of H we have gotten. We're interested in the expected time until absorption, given that we start at ϕ.

$$I - Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & -2/3 & 0 \\ -1/3 & 1 & -2/3 \\ -1/3 & 0 & 1 \end{pmatrix}, \quad F = (I - Q)^{-1} = \begin{array}{c|c} & \begin{array}{ccc} \phi & H & HH \end{array} \\ \begin{array}{c} \phi \\ H \\ HH \end{array} & \begin{pmatrix} 27/8 & 9/4 & 3/2 \\ 15/8 & 9/4 & 3/2 \\ 9/8 & 3/4 & 3/2 \end{pmatrix} \end{array}$$

$$F_{\phi\phi} + F_{\phi H} + F_{\phi HH} = \frac{27}{8} + \frac{9}{4} + \frac{3}{2} = \frac{27 + 18 + 12}{8} = \frac{57}{8} = 7.125 \text{ flips.}$$

Using theory of absorbing chains for studying $\lim_{n\to\infty} P^n$

Consider an example Markov chain where there are two recurrent communication classes R_1 and R_2 whose transition probabilities are described by regular matrices P_1 and P_2 with stationary distributions π_1 and π_2. Suppose there are transient states $T = \{1, 2, \ldots, t\}$. In canonical block form,

$$P = \begin{array}{c} \\ T \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{ccc} T & R_1 & R_2 \end{array} \\ \left(\begin{array}{c|cc} Q & \multicolumn{2}{c}{R} \\ \hline 0 & P_1 & 0 \\ 0 & 0 & P_2 \end{array}\right) \end{array}$$

and

$$\lim_{n\to\infty} P^n = \begin{array}{c} \\ T \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{ccc} T & R_1 & R_2 \end{array} \\ \begin{pmatrix} 0 & * & * \\ 0 & \Lambda_1 & 0 \\ 0 & 0 & \Lambda_2 \end{pmatrix} \end{array}$$

where Λ_1 is a matrix with equal rows given by π_1 and Λ_2 is a matrix with equal rows given by π_2.

We can determine T to R_1 and T to R_2 long-term transition probabilities as follows.

Consider the absorbing chain given by

$$\mathcal{A} = \begin{array}{cc} & \begin{array}{ccc} T & R_1 & R_2 \end{array} \\ \begin{array}{c} T \\ R_1 \\ R_2 \end{array} & \left(\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array} \right) \end{array}$$

where P_1 and P_2 are replaced by I in P

Then $$\lim_{n \to \infty} \mathcal{A}^n = \begin{array}{cc} & \begin{array}{ccc} T & R_1 & R_2 \end{array} \\ \begin{array}{c} T \\ R_1 \\ R_2 \end{array} & \left(\begin{array}{c|c} 0 & (I-Q)^{-1}R \\ \hline 0 & I \end{array} \right) \end{array}$$ and

$(I-Q)^{-1}R$ gives the long term probabilities of being absorbed in states in R_1 and R_2.

For each $i \in T$, let $p_i = \sum_{j \in R_1} [(I-Q)^{-1}R]_{ij}$

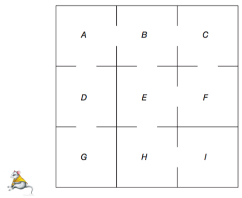
Then $$\lim_{n \to \infty} P^n = \begin{array}{cc} & \begin{array}{ccc} T & R_1 & R_2 \end{array} \\ \begin{array}{c} T \\ R_1 \\ R_2 \end{array} & \left(\begin{array}{ccc} 0 & \tilde{\Lambda}_1 & \tilde{\Lambda}_2 \\ 0 & \Lambda_1 & 0 \\ 0 & 0 & \Lambda_2 \end{array} \right) \end{array}$$

where $$\tilde{\Lambda}_1 = \begin{array}{c} 1 \\ 2 \\ \vdots \\ t \end{array} \left(\begin{array}{c} - \; p_1 \pi_1 \; - \\ - \; p_2 \pi_1 \; - \\ \\ - \; p_t \pi_1 \; - \end{array} \right)$$ and

$$\tilde{\Lambda}_2 = \begin{array}{c} 1 \\ 2 \\ \vdots \\ t \end{array} \left(\begin{array}{c} - \; (1-p_1)\pi_2 \; - \\ - \; (1-p_2)\pi_2 \; - \\ \vdots \\ - \; (1-p_t)\pi_2 \; - \end{array} \right)$$

Can you given an informal explanation why?

Problem 1. Like in our last worksheet, a mouse is placed in the maze below, starting in room A. The trap is placed in room F and the piece of cheese is placed in room I. The nearest neighbor random walk dynamic from room to room remains. What is the expected number of times the mouse visits room A before it either finds the cheese or gets trapped? Room B? What is the expected number steps (ie. rooms visited, counting repitition) before the mouse either finds the cheese or gets trapped?



$$P = \begin{array}{c|ccccccc|cc} & A & B & C & D & E & G & H & F & I \\ \hline A & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ B & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ C & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ D & 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ E & 0 & 1/3 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 \\ G & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ H & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ \hline F & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Let Q be the upper right 7×7 submatrix of P and $F = (I-Q)^{-1}$.

Then ① F_{AA}

② F_{AB}

③ sum of row A of F

```{r}
Q = matrix(c(0, 1/2, 0, 1/2, 0, 0, 0,
             1/3, 0, 1/3, 0, 1/3, 0, 0,
             0, 1/2, 0, 0, 0, 0, 0,
             1/2, 0, 0, 0, 0, 1/2, 0,
             0, 1/3, 0, 0, 0, 0, 1/3,
             0, 0, 0, 1, 0, 0, 0,
             0, 0, 0, 0, 1/2, 0, 0), nrow = 7, ncol = 7, byrow = T)
colnames(Q) =  c("A", "B", "C", "D", "E", "G", "H")
rownames(Q) = colnames(Q)
R = matrix(c(0, 0,
             0, 0,
             1/2, 0,
             0, 0,
             1/3, 0,
             0, 0,
             0, 1/2), nrow = 7, ncol = 2, byrow = T)
rownames(R) = colnames(Q)
colnames(R) = c("F", "I")
solve(eye(7) - Q)["A", "A"] # F_AA gives expected number of visits to A before absorption given X_0 = A
solve(eye(7) - Q)["A", "B"] # F_AB gives expected number of visits to B before absorption given X_0 = A
sum(solve(eye(7) - Q)["A", ]) # sum over the Ath row of (I-Q)^(-1)
```

```
[1] 3.818182
[1] 2.727273
[1] 14.63636
```

Problem 2. A biased coin with heads probability 2/3 is repeatedly flipped. Find the expected number of flips made until the pattern HTHTH first appears.

$$P = \begin{array}{c|ccccc|c} & \emptyset & H & HT & HTH & HTHT & HTHTH \\ \hline \emptyset & 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ H & 0 & 2/3 & 1/3 & 0 & 0 & 0 \\ HT & 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ HTH & 0 & 2/3 & 0 & 0 & 1/3 & 0 \\ HTHT & 1/3 & 0 & 0 & 0 & 0 & 2/3 \\ \hline HTHTH & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Problem 2

```{r}
Q = matrix(c(1/3, 2/3, 0, 0, 0,
             0, 2/3, 1/3, 0, 0,
             1/3, 0, 0, 2/3, 0,
             0, 2/3, 0, 0, 1/3,
             1/3, 0, 0, 0, 0), nrow = 5, ncol = 5, byrow = T)
colnames(Q) = c("empty", "H", "HT", "HTH", "HTHT")
rownames(Q) = colnames(Q)
sum(solve(eye(5)-Q)["empty",])
```

```
[1] 38.625
```

Let Q be the 5×5 upper right block of P and $F = (I-Q)^{-1}$. Then the expected number of flips is the sum of the first row of F, which is 38.625.