6.1 Intro to Poisson process

Recall Poisson rv counts number of occurrences or "arrivals" of event that happens with mean rate λ over fixed time interval. $N \sim Pois(\lambda)$ means $P(N=k) = e^{-\lambda} \frac{\lambda^k}{k!}$ $k = 0, 1, 2, \ldots$

Example On average 3.5 car accidents occur in a town each month. What is the prob. of

1) exactly 4 accidents next month

$N_1 \sim Pois(\lambda)$ is number of accidents over 1 month, $\lambda = 3.5$

$P(N_1=4) = e^{-3.5} \frac{(3.5)^4}{4!} = 0.1888123$

R command: dpois(4, 3.5)

2) at least 2 accidents next month

$P(N_1 \geq 2) = 1 - P(N_1 \leq 1) = 1 - e^{-3.5} \frac{(3.5)^0}{0!} - e^{-3.5} \frac{(3.5)^1}{1!} = 0.8641118$

R command: 1 - ppois(1, 3.5)

3) at most 3 accidents next month

$P(N_1 \leq 3) = P(N_1=0) + P(N_1=1) + P(N_1=2) + P(N_1=3) = 0.5366327$

R command: ppois(3, 3.5)

What if we ask same questions but over 2 month period?

Let $N_2 \sim Pois(7)$ be number of accidents over 2 months.

Basic intuition $Pois(\lambda t)$ counts arrivals in time interval $[0, t]$

R commands: if $N \sim Pois(\lambda)$,

$P(N=k)$ dpois(k, λ)

$P(N \le k)$ ppois(k, λ)

Poisson process models arrivals over <u>variable</u> time interval $[0,t]$ where $t>0$.

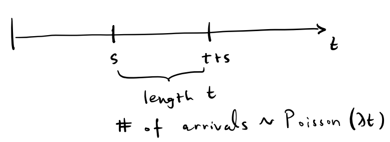
<u>Def</u> A Poisson process with parameter λ is a counting process $(N_t)_{t\ge 0}$ (means N_t is integer valued and increases with time t) such that

1) $N_0 = 0$

2) For all $t>0$, $N_t \sim Poisson(\lambda t)$

3) (stationary increments) For all $s,t>0$

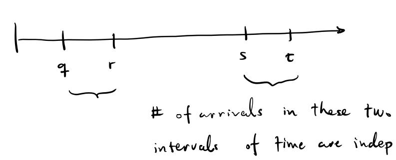
$$N_{t+s} - N_s \sim N_t \sim Poisson(\lambda t)$$



4) (independent increments)

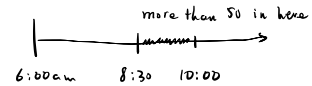
For $0 \le q < r \le s < t$,

$N_t - N_s$ and $N_r - N_q$ are independent



Example Suppose customers arrive at a bakery at a rate of 30 customers per hour according to a Poisson process, starting at 6 a.m. today.

① Find prob. that more than 50 arrive between 8:30 and 10:00 a.m.



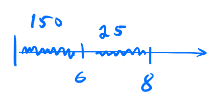
$$P(N_4 - N_{2.5} > 50)$$

$$\overset{\text{Stationary incr.}}{=} P(N_{1.5} > 50) \qquad N_{1.5} \sim Pois(1.5 \cdot 30)$$

1 − ppois(50, 1.5*30)

② Find prob. of getting 150 customers by 12:00pm (noon) and 175 by 2:00 pm.

$$P(N_6 = 150,\ N_8 = 175)$$



$$= P(N_6 = 150,\ N_8 - N_6 = 25)$$

$$\overset{\text{indep. increm.}}{=} P(N_6 = 150)\, P(N_2 = 25)$$

dpois(150, 6*30) * dpois(25, 2*30)

③ Given that 60 people arrive by 8:00 am, find prob. that more than 100 total have arrived by 9:00am.

$$P(N_3 > 100 \mid N_2 = 60) = P(N_1 > 40)$$

1− ppois(40, 30)

Problem 1. Let (N_t) be a Poisson process with mean rate $\lambda = 1.5$. Compute the following probabilities, which are meant to get you familiar with notation and the notions of stationary, independent increments. If the process represents the arrival of emails, with time units of hours, what is the practical meaning of each event described?

a. $P(N_{2.5} = 2, N_5 = 4)$

b. $P(N_5 = 7 \mid N_2 = 3)$

c. $P(N_1 = 2 \mid N_4 = 6)$

Problem 1

```{r}
dpois(2,2.5*1.5)^2
dpois(4,3*1.5)
dpois(4,3*1.5)*dpois(2,1.5)/dpois(6,4*1.5)
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[1] 0.02734365
[1] 0.1898076
[1] 0.2966309
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(a) $P(N_{2.5} = 2, N_5 = 4)$

$= P(N_{2.5} = 2, N_5 - N_{2.5} = 2)$

$= P(N_{2.5} = 2)\, P(N_5 - N_{2.5} = 2)$

$= P(N_{2.5} = 2)^2$

(b) $P(N_5 = 7 \mid N_2 = 3)$

$= \dfrac{P(N_5 = 7, N_2 = 3)}{P(N_2 = 3)}$

$= \dfrac{P(N_5 - N_2 = 4, N_2 = 3)}{P(N_2 = 3)}$

$= P(N_5 - N_2 = 4)$

$= P(N_3 = 4)$

(c) $P(N_1 = 2 \mid N_4 = 6)$

$= \dfrac{P(N_4 = 6 \mid N_1 = 2)\, P(N_1 = 2)}{P(N_4 = 6)}$

$= \dfrac{P(N_3 = 4)\, P(N_1 = 2)}{P(N_4 = 6)}$

Problem 2. Calls are received at a company call center according to a Poisson process at the mean rate of 2.7 per minute. Express the following events in terms of Poisson process N_t notation and then find the probability of the event.

a. No call occurs over a 1.5 minute period.

b. Two calls occur in the first minute, and 4 calls occur in the following two minutes.

c. 11 calls are received in the first three minutes and 6 of those calls occur in the first minute.

Problem 2

```{r}
lambda = 2.7
dpois(0,1.5*lambda)
dpois(2,lambda)*dpois(4,2*lambda)
dpois(5,2*lambda)*dpois(6,lambda)
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[1] 0.01742237
[1] 0.03919909
[1] 0.006249602
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(a) $P(N_{1.5} = 0)$

(b) $P(N_1 = 2, N_3 = 6) = P(N_1 = 2)\, P(N_3 - N_1 = 4)$

$= P(N_1 = 2)\, P(N_2 = 4)$

(c) $P(N_3 = 11, N_1 = 6) = P(N_3 - N_1 = 5)\, P(N_1 = 6)$

$= P(N_2 = 5)\, P(N_1 = 6)$