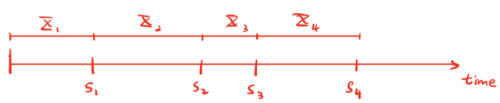
§6.2 Arrival and inter-arrival times

Recall a random variable X has the exponential distribution with parameter $\lambda > 0$ if it's a continuous random variable whose range is $(0, \infty)$ and whose density is $f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$ and cumulative distribution function is $F(x) = P(X \le x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$.

This distribution models waiting times until an event occurs.

Theorem Let $(N_t)_{t \ge 0}$ be a Poisson process with mean arrival rate $\lambda > 0$. Let $X_1, X_2, \ldots$ be the inter-arrival times of (N_t). Then $X_1, X_2, \ldots \sim Exp(\lambda)$ are i.i.d.



Proof Let X_1 be the first arrival time.

Then $P(X_1 > t) = P(\text{no arrivals in } [0, t])$

$$= P(N_t = 0)$$

$$= e^{-\lambda t}$$

Therefore $F_{X_1}(t) = P(X_1 \le t) = 1 - e^{-\lambda t}$ and so X_1 has the cumulative distribution function of the $Exp(\lambda)$ distribution. Let X_2 be the inter-arrival time between the first and second arrivals. Then

$$P(X_2 > t \mid X_1 = s) = P(\text{no arrivals in } [s, s+t])$$

$$= P(N_{s+t} - N_s = 0)$$

$$= P(N_t = 0) \quad (\text{stationary increments})$$

$$= e^{-\lambda t}.$$

Therefore, $X_2 \sim Exp(\lambda)$ and X_2 is independent of X_1.

Proceeding similarly for X_3, X_4, and so on gives the result.

Corollary Let $S_1, S_2, S_3, \ldots$ be the arrival times of the Poisson process. Then $S_n \sim Gamma(n, \lambda)$ for each $n \geq 1$.

Proof Note $S_n = X_1 + X_2 + \cdots + X_n$ and the i.i.d. sum of n exponentially distributed random variabl is Gamma distributed.

Remarks ① if $X \sim Exp(\lambda)$, then $E[X] = \frac{1}{\lambda}$ and $V(X) = \frac{1}{\lambda^2}$.

② Notice the units of λ are arrivals per time unit, and so the units of $1/\lambda$ are time units per arrival.

③ if $S_n \sim Gamma(n, \lambda)$, then $E[S_n] = n/\lambda$ and $V(S_n) = n/\lambda^2$.

④ $X \sim Exp(\lambda)$ has the memoryless property:

$$\underbrace{P(X > t+s \mid X > s)}_{\text{prob. of waiting at least } t \text{ more time units given that you've already waited at least } s \text{ time units}} = \underbrace{P(X > t)}_{\text{prob. of waiting at least } t \text{ time units}}$$

R commands: $X \sim Exp(\lambda)$, $P(X \leq t)$ pexp(t, λ)

$S \sim Gamma(n, \lambda)$ $P(S \leq t)$ pgamma(t, n, λ)

Problem 1. Customers arrive at a store starting at 6:00 am according to a Poisson process with a mean rate of 8 customers per hour. Each of the following events can be expressed in two ways: (a) in terms of N_t, (b) in terms of arrival or inter-arrival times. For each event, use whichever is easier and then use R to compute the probability of the event. If you have time, you can try both methods and check that your answers agree.

a. No customer comes in the first hour of the store opening.

b. There is at least 30 minutes between the arrivals of the first and second customers.

c. There is at least 2 hours between the arrivals of the third and fifth customers.

d. The 50th customer comes in after 1:00 pm.

e. In the first 5 hours, at least 40 customers come in.

Problem 1

```{r}
lambda = 8
dpois(0,lambda); 1-pexp(1,lambda)
dpois(0,0.5*lambda); 1-pexp(0.5,lambda)
ppois(1,2*lambda); 1-pgamma(2,2,lambda)
ppois(49,7*lambda); 1-pgamma(7,50,lambda)
1-ppois(39,5*lambda); pgamma(5,40,lambda)
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[1] 0.0003354626
[1] 0.0003354626
[1] 0.01831564
[1] 0.01831564
[1] 1.913098e-06
[1] 1.913098e-06
[1] 0.193934
[1] 0.193934
[1] 0.5210289
[1] 0.5210289
```

(a) $P(N_1 = 0) = P(X_1 > 1)$

(b) $P(X_2 > 1/2) = P(N_{0.5} = 0)$

(c) $P(S_5 - S_3 > 2) = P(X_5 + X_4 > 2)$

$= P(S_2 > 2)$

$= P(N_2 \leq 1)$

(d) $P(S_{50} > 7) = P(N_7 \leq 49)$

(e) $P(N_5 \geq 40) = P(S_{40} \leq 5)$

Problem 2. At a certain intersection, red, green, and orange cars go by according to three independent Poisson processes. On average there is one red car every 10 minutes, one green car every 15 minutes, and one orange car every 20 minutes.

a. What are the parameters of the Poisson processes, using minutes as time units?

b. What is the probability that you wait at least 15 minutes between seeing your fourth and fifth orange car pass?

c. What is the probability that the third green car you see goes by between minutes 25 and 30 of you watching?

d. You've already waited 40 minutes for a red car without seeing one. Given this, what is the probability you'll need to wait at least an additional 15 minutes before seeing a red car? What is the expected additional time you'll wait for a red car?

e. Bonus questions that preview some ideas to come next time:

1. What is the probability of waiting more than 10 minutes for a car, of any color, to arrive? How long will you wait on average before a car, of any color, arrives?[1]
2. What is the probability that the first car to arrive is green?[2]

Problem 2

```{r}
lambda_R = 1/10; lambda_G = 1/15; lambda_O = 1/20
1-pexp(15,lambda_O)
pgamma(30,3,lambda_G) - pgamma(25,3,lambda_G)
1-pexp(15,lambda_R)
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```
[1] 0.4723666
[1] 0.08931908
[1] 0.2231302
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(a) $\lambda_R = 1/10, \quad \lambda_G = 1/15, \quad \lambda_O = 1/20$

(b) $P(S_5 - S_4 > 15) = P(X_5 > 15), \quad X_5 \sim Exp(\lambda_O)$

(c) $P(25 < S_3 < 30), \quad S_3 \sim Gamma(3, \lambda_G)$

(d) $P(X > 55 \mid X > 40) = P(X > 15), \quad X \sim Exp(\lambda_R)$

$E[\text{additional wait time}] = E[X] = 1/\lambda_R = 10$

(e) (1) method 1: Let $(N_t)_{t\geq 0}$ be a Poisson process with rate $\lambda = \lambda_R + \lambda_G + \lambda_O$ which models arrivals of cars of any of the three colors.

Then $P(N_{10}=0)$ where $N_{10} \sim Pois(10\lambda)$

method 2: Let $M = \min\{X_R, X_G, X_O\}$

Then
$$P(M>t) = P(X_R>t, X_G>t, X_O>t)$$
$$= P(X_R>t)\,P(X_G>t)\,P(X_O>t)$$
$$= e^{-\lambda_R t}\, e^{-\lambda_G t}\, e^{-\lambda_O t}$$
$$= e^{-(\lambda_R+\lambda_G+\lambda_O)t}$$

which means $M \sim Exp(\lambda_R+\lambda_G+\lambda_O)$. Therefore we can compute $P(M>10) = e^{-(1/10+1/15+1/20)10}$

(2)
$$P(M = X_G) = P(\min\{X_R, X_G, X_O\} = X_G)$$
$$= P(X_R \geq X_G,\ X_O \geq X_G)$$
$$= \int_0^\infty P(X_R \geq X_G, X_O \geq X_G \mid X_G = t)\,\lambda_G e^{-\lambda_G t}\,dt$$
$$= \int_0^\infty P(X_R \geq t, X_O \geq t)\,\lambda_G e^{-\lambda_G t}\,dt$$
$$= \int_0^\infty e^{-\lambda_R t}\, e^{-\lambda_O t}\,\lambda_G e^{-\lambda_G t}\,dt$$
$$= \lambda_G \int_0^\infty e^{-(\lambda_R+\lambda_G+\lambda_O)t}\,dt$$
$$= \frac{\lambda_G}{\lambda_R+\lambda_G+\lambda_O}$$
$$= \frac{1/15}{1/10+1/15+1/20}$$