§6.4 Thinning and Superposition

Theorem Let $X_1, X_2, \ldots, X_n$ be independent, exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Let $M = \min\{X_1, \ldots, X_n\}$.

Then ① $M \sim Exp(\lambda_1 + \cdots + \lambda_n)$

② $P(M = X_k) = \dfrac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}$

Proof Same as bonus problems on worksheet from last time.

Def Let $(N_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$ be independent Poisson processes. Then $(N_t + M_t)_{t\geq 0}$ is called the superposition of $(N_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$.

Theorem If $(N_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$ are independent Poisson process with mean rates λ and μ, then their superposition is a Poisson process with mean rate $\lambda+\mu$.

Core idea independent sums of Poisson random variables are Poisson distributed with mean rate given by sum of mean rates.

Def Suppose $(N_t)_{t\geq 0}$ is a Poisson process with mean rate λ, and each arrival is, independently, of type I with probability p or type II with prob. $1-p$. Let $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ be the counting processes for type I and type II arrivals. These are called thinned Poisson processes.

Lemma In the context above, $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are independent with mean rates $p\lambda$ and $(1-p)\lambda$ respectively.

Core idea Observe that

$$
\begin{aligned}
P(A_t = a, B_t = b) &= P(A_t = a, N_t = a+b) \\
&= P(A_t = a \mid N_t = a+b) P(N_t = a+b) \\
&= \binom{a+b}{a} p^a (1-p)^b \cdot e^{-\lambda t} \frac{(\lambda t)^{a+b}}{(a+b)!} \\
&= \frac{(a+b)!}{a!\,b!} p^a (1-p)^b \cdot e^{-\lambda t} \frac{(\lambda t)^{a+b}}{(a+b)!} \\
&= e^{-\lambda t} \frac{(p\lambda t)^a}{a!} \frac{((1-p)\lambda t)^b}{b!} \\
&= e^{-p\lambda t} \frac{(p\lambda t)^a}{a!} \, e^{-(1-p)\lambda t} \frac{((1-p)\lambda t)^b}{b!}
\end{aligned}
$$

Problem 1. Starting at 6 a.m., cars, buses, and motorcycles arrive at a highway toll booth according to independent Poisson processes. Cars arrive about once every 5 minutes. Buses arrive about once every 10 minutes. Motorcycles arrive about once every 30 minutes. Let $(C_t)_{t\geq 0}, (B_t)_{t\geq 0}, (M_t)_{t\geq 0}$, and $(V_t)_{t\geq 0}$ denote the Poisson processes for cars, buses, motorcycles, and vehicles of any of these three types respectively, and let $\lambda_C, \lambda_B, \lambda_M, \lambda_V$ denote their mean rates.

a. State the values of $\lambda_C, \lambda_B, \lambda_M, \lambda_V$.

b. Find the probability that in the first 20 minutes exactly three vehicles arrive—two buses and one motorcycle.

c. Find the probability that in the first 20 minutes exactly three vehicles arrive.

d. Find the probability that the first vehicle to arrive is a car.

e. At the toll booth, the chance that a driver has exact change is 1/4, independent of vehicle. Find the probability that no vehicle has exact change in the first 10 minutes.

f. Find the probability that it takes at least 20 minutes for 5 vehicles with exact change to arrive.

g. Find the probability that the seventh motorcycle arrives within 45 minutes of the third motorcycle.

(a) $\lambda_C = 1/5, \quad \lambda_B = 1/10, \quad \lambda_M = 1/30, \quad \lambda_V = \lambda_C + \lambda_B + \lambda_M$

(b) $P(C_{20} = 0, B_{20} = 2, M_{20} = 1) = P(C_{20} = 0)\, P(B_{20} = 2)\, P(M_{20} = 1)$

(c) $P(V_{20} = 3)$

(d) $\dfrac{\lambda_C}{\lambda_C + \lambda_B + \lambda_M}$

(e) Let $(E_t)_{t\geq 0}$ be the thinning of $(V_t)_{t\geq 0}$ with mean rate $\frac{1}{4}\lambda_V$.

$P(E_{10} = 0)$

(f) Let $S_5 \sim Gamma(5, 1/4\, \lambda_V)$.

$P(S_5 \geq 20)$

(g) Let $S_n \sim Gamma(n, \lambda_M)$

$P(S_7 - S_3 \leq 45) = P(S_4 \leq 45)$

Problem 1

```{r}
lambda_c = 1/15; lambda_b = 1/10; lambda_m = 1/30
lambda_v = lambda_c + lambda_b + lambda_m
dpois(0, 20*lambda_c)*dpois(2, 20*lambda_b)*dpois(1,20*lambda_m) # part b
dpois(3, 20*lambda_v) # part c
lambda_c/(lambda_v) # part d
dpois(0, 10*1/4*lambda_v) # part e
1-pgamma(20,5,1/4*lambda_v) # part f
pgamma(45,4,lambda_m) # part g
```

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[1] 0.02442085
[1] 0.1953668
[1] 0.3333333
[1] 0.6065307
[1] 0.9963402
[1] 0.06564245
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