§ 6.3 Infinitesimal probabilities

Let $(N_t)_{t \geq 0}$ be a Poisson process with mean rate λ.

Question If we consider a small time interval, say $[0, \Delta t]$ of size Δt, how many arrivals are possible within this time interval?

① $P(N_{\Delta t} = 0) = e^{-\lambda \Delta t} = 1 - \lambda \Delta t + \frac{(-\lambda \Delta t)^2}{2!} + \cdots$

$= 1 - \lambda \Delta t +$ terms of order Δt^2 or higher

② $P(N_{\Delta t} = 1) = e^{-\lambda \Delta t} \cdot \lambda \Delta t$

$= \left(1 - \lambda \Delta t + \frac{(-\lambda \Delta t)^2}{2!} + \cdots\right) \lambda \Delta t$

$= \lambda \Delta t +$ terms of order Δt^2 or higher

③ $P(N_{\Delta t} \geq 2) = 1 - P(N_{\Delta t} \leq 1)$

$=$ terms of order Δt^2 or higher

Intuition Having 2 or more arrivals occur within the time interval $[0, \Delta t]$ is an order of magnitude less likely than 1 arrival. So at an infinitesimal time scale, only 0 or 1 arrivals are possible.

§7.1 Continuous-time Markov Chains (CTMC)

Def Let $(X_t)_{t\geq 0}$ be a continuous-time stochastic process with countable state space S. Then $(X_t)_{t\geq 0}$ is a continuous-time Markov chain if

$$P(X_{t+s}=j \mid X_s=i,\ X_u=x_u,\ 0\leq u<s) = P(X_{t+s}=j \mid X_s=i)$$

for any $i,j,x_u \in S$ and $s,t \in [0,\infty)$. It is called time-homogeneous if $P(X_{t+s}=j \mid X_s=i) = P(X_t=j \mid X_0=i)$.

The function $P:[0,\infty) \to \underbrace{\mathbb{R}^{|S|\times|S|}}_{\text{set of } |S|\times|S| \text{ matrices}}$ given by

$P_{ij}(t) = P(X_t=j \mid X_0=i)$ is called the transition function.

How to think about the dynamic of a CTMC:

1. wait a random amount of time to transition
2. then transition to a new state randomly.

Lemma Let $i \in S$ and let T_i be the holding time at i (ie. the time until we transition to a new state). Then T_i must be exponentially distributed (with some parameter which we can denote as q_i).

Proof Since the exponential distribution is the only continuous distribution with the memoryless property, it suffices to show that T_i has the memoryless property. Observe that

$$P(T_i > t+s \mid T_i > s) = P(X_{t+s} = i \mid X_u = i,\ 0 \le u \le s)$$

$$= P(X_{t+s} = i \mid X_s = i)$$

$$= P(X_t = i)$$

$$= P(T_i > t)$$

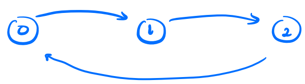
When the chain leaves i, it goes to a new state $j \ne i$, determined by some probability $\tilde{P}_{ij}$. The sequence of states visited $Y_0, Y_1, Y_2, \ldots$ is a discrete-time Markov chain called the embedded chain with transition matrix $\tilde{P}$ where $\tilde{P}_{ii} = 0$, $\sum_{j \ne i} \tilde{P}_{ij} = 1$.

Example A Poisson process $(N_t)_{t \ge 0}$ with mean rate λ is a CTMC with state space $\{0, 1, 2, \ldots\}$, hold times $T_i \sim \mathrm{Exp}(\lambda)$, and embedded chain transition matrix

$$\tilde{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 \cdots \\ 0 & 0 & 0 & 1 \cdots \\ & & & \end{pmatrix} \end{array}$$

Problem 1. Consider a hair salon with two chairs—chair 1 and chair 2. A customer upon arrival goes initially to chair 1 where their hair is shampooed and rinsed. After this is done, the customer moves on to chair 2 where their hair is cut. The service times at the two chairs are assumed to be independent random variables that are exponentially distributed with respective rates μ_1 and μ_2. Suppose that potential customers arrive in accordance with a Poisson process with rate λ, and a potential customer will enter the system only if both chairs are empty. This is a CTMC with the following states.

State	Interpretation
0	salon is empty
1	a customer is in chair 1
2	a customer is in chair 2



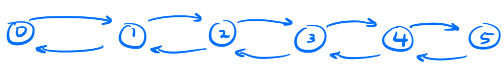
$$q_0 = \lambda$$

$$q_1 = \mu_1$$

$$q_2 = \mu_2$$

$$\tilde{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

Problem 2. At a small shop, there is one cash register where customers can check out, one at a time. Customers arrive in line to check out according a Poisson process with rate λ. The service time to check out each customer is independent from customer to customer and exponentially distributed with parameter μ. If the line is full (ie. there are 5 customers in line or checking out) an arriving customer will simply not get in line. This is a CTMC where the state space $\mathcal{S} = \{0, 1, 2, 3, 4, 5\}$ represents the number of customers in line or checking out.



$$q_0 = \lambda$$

$$q_i = \lambda + \mu \quad , \quad i = 1, 2, 3, 4$$

$$q_5 = \mu$$

$$\tilde{P} = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 & 0 \\ 2 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 \\ 3 & 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 \\ 4 & 0 & 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$