§7.2 Alarm clocks and transition rates

We begin with another example of a CTMC.

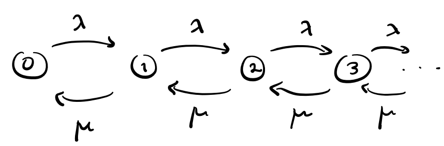
memoryless arrivals

single server

<u>Example</u> In an $M/M/1$ queue, customers

memoryless service times

arrive in line at a mean rate λ according to a Poisson process and are processed by a single server, one at a time with exponential service times with mean rate μ. Let X_t be the number of customers at time $t \geq 0$.



If $X_t = i$, we can only transition to state $i+1$ (the queue grows) or state $i-1$ (the queue shrinks). Who wins: the arrival process or the server process?

The hold time $T_i = \min\{X, Y\}$ where

competing independent "alarm clocks"

$X \sim \text{Exp}(\lambda)$ is the time until next customer arrival

$Y \sim \text{Exp}(\mu)$ is the service time of current customer.

From previous theory, we know

① $T_i \sim Exp(q_i)$ where $q_i = \lambda + \mu$

② $\tilde{P}_{i,i+1} = P(T_i = X) = \dfrac{\lambda}{\lambda+\mu}$

③ $\tilde{P}_{i,i-1} = P(T_i = Y) = \dfrac{\mu}{\lambda+\mu}$.

__Def__ The average rate at which transitions occur from i to j is called the __transition rate__ q_{ij}.

__Facts__ if it's possible to transition from state i to states $1, \dots, k$ with transition rates $q_{i1}, \dots, q_{ik}$

then ① $T_i \sim Exp(q_i)$ where $q_i = q_{i1} + \cdots + q_{ik}$

② $\tilde{P}_{ij} = \dfrac{q_{ij}}{q_i}$

So given the transition rates of a CTMC, we can determine the hold times and embedded chain transition probabilities.

For each of the following examples of continuous time Markov chains (CTMC):

a. Draw a transition diagram showing the possible transitions that can occur between states. Label the edges with transition rates.

b. Find the hold-time parameter for each state i.

c. Find the transition matrix $\widetilde{P}$ for the embedded chain.

Example 1. Consider two independent machines that are maintained by a single person. Each machine functions for an exponentially distributed amount of time before breaking down with an average time of 3 hours between breakdowns. The repair time for either machine is exponentially distributed with an average repair time of 4 hours. Let X_t denote the number of broken machines at time t in hours.

$$\begin{tikzcd} 0 \arrow[r, bend left, "2/3"] & 1 \arrow[l, bend left, "1/4"] \arrow[r, bend left, "1/3"] & 2 \arrow[l, bend left, "1/4"] \end{tikzcd}$$

$q_0 = 2/3$

$q_1 = 1/3 + 1/4 = 7/12$

$q_2 = 1/4$

$$\widetilde{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 3/7 & 0 & 4/7 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Example 2. Suppose in the previous example that there are 2 maintenance people. The time it takes either of them, working alone, to repair a machine is exponentially distributed with an average repair time of 4 hours. Suppose that if only one machine is broken, one of them repairs it and the other is idle. If two machines are broken, then they can work simultaneously, but independently, on each machine. Let X_t denote the number of broken machines at time t in hours.

$$\begin{tikzcd} 0 \arrow[r, bend left, "2/3"] & 1 \arrow[l, bend left, "1/4"] \arrow[r, bend left, "1/3"] & 2 \arrow[l, bend left, "2/4"] \end{tikzcd}$$

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Example 3. Consider a population where members produce offspring but cannot die. Suppose each member acts independently and takes an exponentially distributed amount of time, on average 6 months, to produce an offspring. Let X_t be the population size at time t in years and suppose that $X_0 > 0$.

$$\begin{tikzcd} 1 \arrow[r, "2"] & 2 \arrow[r, "4"] & 3 \arrow[r, "6"] & 4 \arrow[r, "8"] & \cdots \end{tikzcd}$$

$$\widetilde{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & \cdots \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ 0 & 0 & 0 & 1 & \cdots \\ & & & & \end{pmatrix} \end{array}$$

$q_i = 2i$ for all $i \geq 1$

Example 4. Like in the last example, consider a population where each member acts independently and takes an exponentially distributed amount of time, on average 6 months, to produce an offspring. Further, suppose that the lifespan of each member is exponentially distributed, with an average lifespan of 4 years. is Let X_t be the population size at time t in years.

$$\begin{tikzcd} 0 \arrow[r, bend left, "0"] & 1 \arrow[l, bend left, "1/4"] \arrow[r, bend left, "2"] & 2 \arrow[l, bend left, "2/4"] \arrow[r, bend left, "4"] & 3 \arrow[l, bend left, "3/4"] \arrow[r, bend left, "6"] & 4 \arrow[l, bend left, "1"] \arrow[r, bend left, "8"] & \cdots \arrow[l, bend left, "5/4"] \end{tikzcd}$$

$q_0 = 0$

$q_i = 2i + i/4 = 9i/4$

$$\widetilde{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1/9 & 0 & 8/9 & 0 & \cdots \\ 0 & 1/9 & 0 & 8/9 & \cdots \\ 0 & 0 & 1/9 & 0 & 8/9 \cdots \\ & & & & \end{pmatrix} \end{array}$$