

§ 7.3 Infinitesimal generator

Recall that $P_{ij}(t) = P(\bar{X}_t = j | \bar{X}_0 = i)$ is

the transition function of a CTMC $(\bar{X}_t)_{t \geq 0}$.

Our goal today is to understand how to compute $P(t)$.

Theorem (Kolmogorov forward equation) The transition function $P(t)$ satisfies the differential equation

$$P'(t) = P(t)Q$$

where $Q = P'(0)$.

Proof Note that for any $t, h > 0$,

$$\begin{aligned} P_{ij}(t+h) &= P(\bar{X}_{t+h} = j | \bar{X}_0 = i) \\ &= \sum_{k \in S} P(\bar{X}_{t+h} = j | \bar{X}_t = k) P(\bar{X}_t = k | \bar{X}_0 = i) \\ &= \sum_{k \in S} P_{ik}(t) P_{kj}(h) \\ &= [P(t)P(h)]_{ij} \end{aligned}$$

Therefore, $P(t+h) = P(t)P(h)$ (this is called

the Chapman-Kolmogorov equation). Next, note that

$$\begin{aligned} P'(t) &= \lim_{h \rightarrow 0^+} \frac{P(t+h) - P(t)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{P(t)P(h) - P(t)}{h} \end{aligned}$$

$$\begin{aligned}
&= P(t) \lim_{h \rightarrow 0^+} \frac{P(h) - I}{h} \\
&= P(t) P'(0) \\
&= P(t) Q.
\end{aligned}$$

Def Given a matrix A , the matrix e^A given

$$\text{by } e^A = I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k$$

is called the matrix exponential.

Theorem $P(t) = e^{tQ}$ solves the Kolmogorov Forward Equation.

Proof Observe that if $P(t) = e^{tQ}$, then

$$\begin{aligned}
P'(t) &= \frac{d}{dt} (e^{tQ}) \\
&= \frac{d}{dt} \left(I + tQ + \frac{t^2}{2!} Q^2 + \frac{t^3}{3!} Q^3 + \dots \right) \\
&= Q + tQ^2 + \frac{t^2}{2!} Q^3 + \frac{t^3}{3!} Q^4 + \dots \\
&= \left(I + tQ + \frac{t^2}{2!} Q^2 + \frac{t^3}{3!} Q^3 + \dots \right) Q \\
&= e^{tQ} Q \\
&= P(t) Q.
\end{aligned}$$

Definition The matrix $Q = P'(0)$ is called the infinitesimal generator of the Markov chain.

Lemma Let Q be the infinitesimal generator and let $i \in S$.

① for each $j \neq i$, Q_{ij} is the mean transition rate from i to j .

② $Q_{ii} = -q_i$, where q_i is the hold time parameter of i .

Proof. ① Let $j \in S$ with $j \neq i$. Then

$$\begin{aligned} Q_{ij} &= P'_{ij}(0) \\ &= \lim_{h \rightarrow 0^+} \frac{P_{ij}(h) - P_{ij}(0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{P_{ij}(h)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{P(\text{1 transition to } j \text{ in } [0, h])}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{E[\# \text{ of transitions to } j \text{ in } [0, h]]}{h} \\ &= q_{ij} \end{aligned}$$

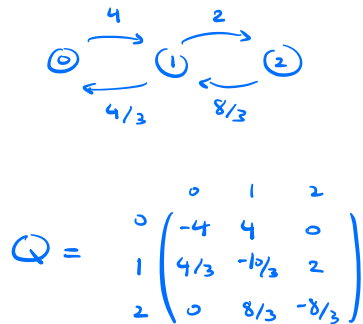
② Observe that

$$\begin{aligned} Q_{ii} &= P'_{ii}(0) \\ &= \lim_{h \rightarrow 0^+} \frac{P_{ii}(h) - P_{ii}(0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{P_{ii}(h) - 1}{h} \\ &= \lim_{h \rightarrow 0^+} -\frac{(1 - P_{ii}(h))}{h} \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0^+} - \frac{\sum_{j \neq i} P_{ij}(h)}{h} \quad (\text{since the row sum is 1}) \\
&= - \sum_{j \neq i} \lim_{h \rightarrow 0^+} \frac{P_{ij}(h)}{h} \\
&= - \sum_{j \neq i} q_{ij} \\
&= -q_i.
\end{aligned}$$

Problem 1. Consider two independent machines that are maintained by a single person. Each machine functions for an exponentially distributed amount of time before breaking down. On average each machine functions for a half hour before breaking down. The repair time for either machine is exponentially distributed. The average repair time is 45 minutes. Assume that at time $t = 0$ (8:00 am) neither machine is broken. Find the following probabilities.

- Both machines are broken at 10:30 am.
- Neither machine is broken at 11:00 am.
- One machine is broken at 2:15 pm.
- The long term probabilities that 0, 1, or 2 machines are broken.



- $P_{02}(2.5)$
- $P_{00}(3)$
- $P_{01}(6.25)$

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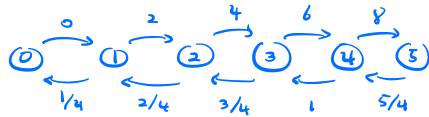
{r}
Q = matrix(c(-4,4,0,
             4/3,-10/3,2,
             0,8/3,-8/3), nrow = 3, ncol = 3, byrow = TRUE)
colnames(Q) = 0:2
rownames(Q) = colnames(Q)
expm(2.5*Q)[0,2] # part a
expm(3*Q)[0,0] # part b
expm(6.25*Q)[0,1] # part c
expm(10*Q)[0,] # part d
# (note t = 10 is long enough to reach equilibrium)
...

[1] 0.3598271
[1] 0.1600218
[1] 0.4800024
      0      1      2
0.1635828 0.4907483 0.3680613

```

Problem 2. Consider a population where each member acts independently and takes an exponentially distributed amount of time, on average 6 months, to produce an offspring. Further, suppose that the lifespan of each member is exponentially distributed, with an average lifespan of 4 years. Finally suppose that when the population size is 5, offspring are no longer produced. Let X_t be the population size at time t in years and suppose that $X_0 = 1$. Find the distribution of the population size and its mean after

- at 3 months
- at 1 year
- at 18 months



$$Q = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & -2.25 & 2 & 0 & 0 & 0 \\ 0 & 1/2 & -4.5 & 4 & 0 & 0 \\ 0 & 0 & 3/4 & -6.75 & 6 & 0 \\ 0 & 0 & 0 & 1 & -9 & 8 \\ 0 & 0 & 0 & 0 & 5/4 & -6/4 \end{pmatrix} \end{matrix}$$

```
## Problem 2
```{r}
Q = matrix(c(0,0,0,0,0,0,
 .25,-2.25,2,0,0,0,
 0,.5,-4.5,4,0,0,
 0,0,.75,-6.75,6,0,
 0,0,0,1,-9,8,
 0,0,0,0,1.25,-1.25), nrow = 6, ncol = 6, byrow = TRUE)
colnames(Q) = 0:5
rownames(Q) = colnames(Q)
expm(0.25*Q)[^2,] # part a distribution
sum(expm(0.25*Q)[^2,]*0:5) # part a mean
expm(1*Q)[^2,] # part b distribution
sum(expm(1*Q)[^2,]*0:5) # part b mean
expm(1.5*Q)[^2,] # part c distribution
sum(expm(1.5*Q)[^2,]*0:5) # part c mean
```
```

```

      0      1      2      3      4      5
0.0023216 0.0563669 0.3638740 0.2722903 0.1594020 0.1457453
[1] 2.96732
      0      1      2      3      4      5
0.0111507 0.0294616 0.0453654 0.0646872 0.1460173 0.7033177
[1] 4.414912
      0      1      2      3      4      5
0.0137232 0.0135619 0.0181260 0.0348326 0.1336155 0.7861409
[1] 4.619478
```