7.4 Long-term behavior

Def Let π be a probability distribution.

Then π is a limiting distribution if

$\lim_{t \to \infty} P(t) = \Lambda$ where Λ's rows are all π.

Further, π is stationary if $\pi P(t) = \pi$ for all $t \geq 0$.

Proposition π stationary iff $\pi Q = 0$.

Proof π stationary $\Rightarrow$ $\pi P(t) = \pi$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} (\pi P(t)) = \left.\frac{d}{dt}\right|_{t=0} (\pi)$$

$$\Rightarrow \pi P'(0) = 0 \Rightarrow \pi Q = 0$$

if $\pi Q = 0$, then $P'(t) = Q P(t)$ (backward equation)

$$\Rightarrow \pi P'(t) = \pi Q P(t)$$

$$= 0$$

$\Rightarrow \pi P(t)$ is constant

$$= \pi P(0) = \pi$$

Question Suppose π is limiting. How to find π exactly using algebra?

Answer We know

$$P'(t) = P(t)\,Q \quad \text{(forward equation)}$$

Take limit as $t \to \infty$ of both sides:

$$\underbrace{\lim_{t\to\infty} P'(t)}_{\substack{=0 \text{ since} \\ P(t) \text{ doesn't} \\ \text{change in long-term}}} = \underbrace{\lim_{t\to\infty} P(t)}_{=\pi}\,Q$$

$$\Longrightarrow \quad 0 = \pi Q$$

$\pi Q = 0$

$\Leftrightarrow Q^T \pi^T = 0^T$

$\left[\, Q^T \,\middle|\, \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \,\right]$

Conclusion When limiting distribution exists, we find it by solving $\pi Q = 0$ equation.

Theorem Let $(X_t)_{t\geq 0}$ be a finite state CTMC which is irreducible (all states communicate). Then it has a unique stationary, limiting distribution.

Remark aperiodicity not an issue in continuous time.