§7.4 Long-term behavior

Def Let $(X_t)_{t\geq 0}$ be a CTMC with state space S. A probability vector π on S is a limiting distribution for (X_t) if $\lim_{t\to\infty} P(t) = \Lambda$ where Λ is a matrix with equal rows consisting of π. Further, π is a stationary distribution for (X_t) if $\pi P(t) = \pi$ for all $t \geq 0$.

Lemma π is a stationary distribution if and only if $\pi Q = 0$ where Q is the infinitesimal generator of (X_t).

Proof $(\Rightarrow)$ Suppose π is stationary. Observe that

$$\begin{aligned}
\pi Q &= \pi P'(0) \\
&= \pi \lim_{h\to 0^+} \frac{P(h) - I}{h} \\
&= \lim_{h\to 0^+} \frac{\pi P(h) - \pi}{h} \\
&= \lim_{h\to 0^+} \frac{\pi - \pi}{h} = 0.
\end{aligned}$$

$(\Leftarrow)$ Suppose $\pi Q = 0$. We must show that $\pi P(t) = \pi$ for all $t \geq 0$. We claim $\pi P(t)$ is constant for all $t \geq 0$. Indeed observe that

$$\frac{d}{dt}\left(\pi P(t)\right) = \pi P'(t)$$

$$= \pi Q P(t) \quad \text{(Kolmogorov Backward Equation)}$$

$$= 0 \, P(t)$$

$$= 0$$

Therefore $\pi P(t)$ is constant in t and so

$$\pi P(t) = \pi P(0) = \pi I = \pi$$

for all $t \geq 0$.

Lemma If π is a limiting distribution, then it's a stationary distribution.

Proof Suppose π is limiting. We'll prove π is stationary by showing $\pi Q = 0$. Observe that for any initial distribution α

$$\pi Q = \lim_{t \to \infty} \alpha P(t) Q$$

$$= \lim_{t \to \infty} \alpha P'(t)$$

$$= \alpha \lim_{t \to \infty} P'(t)$$

$$= \alpha \lim_{t \to \infty} \lim_{h \to 0} \frac{P(t+h) - P(t)}{h}$$

$$= \alpha \lim_{h \to 0} \lim_{t \to \infty} \frac{P(t+h) - P(t)}{h}$$

$$= 0$$

Corollary When limiting distribution π exists, we can compute it by solving $\pi Q = 0$.

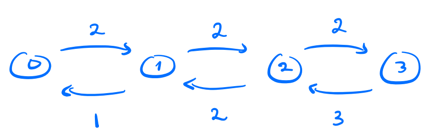
Theorem Let $(X_t)_{t \geq 0}$ be a finite-state CTMC. Then if (X_t) is irreducible (i.e. all states communicate) it has a unique stationary distribution which is a limiting distribution.

Problem 1. During lunch hour, customers arrive at a fast-food restaurant at the rate of 120 customers per hour. The restaurant has one line, with three workers taking food orders at independent service stations. Each worker takes an exponentially distributed amount of time—on average 1 minute—to service a customer. Assume that customers turn away from the store if all three service stations are busy. Let X_t denote the number of service stations busy at time t in minutes.

a. Find the generator Q.

b. Find the limiting distribution π.

c. In the long-term, what is the expected number of busy service stations?



(a)
$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 3 & -3 \end{pmatrix} \end{array}$$

(b) $\pi = \left(\frac{3}{19}, \frac{6}{19}, \frac{6}{19}, \frac{4}{19}\right)$

(c) $\frac{3}{19}(0) + \frac{6}{19}(1) + \frac{6}{19}(2) + \frac{4}{19}(3) = \frac{30}{19}$

Problem 1

```{r}
Q = matrix(c(-2,2,0,0,
             1,-3,2,0,
             0,2,-4,2,
             0,0,3,-3), nrow = 4, ncol = 4, byrow = TRUE)
A = cbind(t(Q), 0) # add column of 0's to Q^t
A = rbind(A, 1) # add row of 1's
fractions(rref(A))
pi = fractions(rref(A))[1:4,5] # limiting distribution
sum(pi*(0:3)) # mean number of busy stations
```

```
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    0    0    0 3/19
[2,]    0    1    0    0 6/19
[3,]    0    0    1    0 6/19
[4,]    0    0    0    1 4/19
[5,]    0    0    0    0    0
[1] 30/19
```

Problem 2. A facility has four machines, with two repair workers to maintain them. Individual machines fail on average every 10 hours. It takes an individual maintenance person on average 4 hours to fix a machine. Repair and failure times are independent and exponentially distributed. Let X_t denote the number of working machines at time t in hours.

a. Find the generator Q.

b. Find the limiting distribution π.

c. In the long-term, what is the expected number of operational machines?

$$\begin{tikzcd} 0 \arrow[r, bend left, "2/4"] & 1 \arrow[l, bend left, "1/10"] \arrow[r, bend left, "2/4"] & 2 \arrow[l, bend left, "2/10"] \arrow[r, bend left, "2/4"] & 3 \arrow[l, bend left, "3/10"] \arrow[r, bend left, "1/4"] & 4 \arrow[l, bend left, "4/10"] \end{tikzcd}$$

(a)

$$Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 1/10 & -3/5 & 1/2 & 0 & 0 \\ 2 & 0 & 1/5 & -7/10 & 1/2 & 0 \\ 3 & 0 & 0 & 3/10 & -11/20 & 1/4 \\ 4 & 0 & 0 & 0 & 2/5 & -2/5 \end{array}$$

(b)

$$\pi = \left(\frac{48}{2513}, \frac{240}{2513}, \frac{600}{2513}, \frac{1000}{2513}, \frac{625}{2513} \right)$$

(c)

$$\frac{48}{2513}(0) + \frac{240}{2513}(1) + \frac{600}{2513}(2) + \frac{1000}{2513}(3) + \frac{625}{2513}(4) = \frac{6940}{2513}$$

Problem 2

```{r}
Q = matrix(c(-1/2,1/2,0,0,0,
             1/10,-3/5,1/2,0,0,
             0,1/5,-7/10,1/2,0,
             0,0,3/10,-11/20,1/4,
             0,0,0,2/5,-2/5), nrow = 5, ncol = 5, byrow = TRUE)
A = cbind(t(Q), 0) # add column of 0's to Q^t
A = rbind(A, 1) # add row of 1's
fractions(rref(A))
pi = fractions(rref(A))[1:5,6] # limiting distribution
sum(pi*(0:4)) # mean number of working machines
```

```
     [,1] [,2] [,3] [,4] [,5]      [,6]
[1,]    1    0    0    0    0   48/2513
[2,]    0    1    0    0    0  240/2513
[3,]    0    0    1    0    0  600/2513
[4,]    0    0    0    1    0 1000/2513
[5,]    0    0    0    0    1  625/2513
[6,]    0    0    0    0    0         0
[1] 6940/2513
```