§ 2.3, 2.4 Joint distribution and long-term behavior

Consider a Markov chain $(X_n)_{n\geq 0}$ with transition matrix P, state space $S = \{1, 2, \ldots, m\}$, and initial distribution α.

Then $$P(X_{n_1} = i,\ X_{n_1+n_2} = j) = (\alpha P^{n_1})_i\ P^{n_2}_{ij}$$

Proof Observe that

$$P(X_{n_1} = i,\ X_{n_1+n_2} = j)$$

$$= P(X_{n_1+n_2} = j \mid X_{n_1} = i)\, P(X_{n_1} = i) \quad (\text{recall } P(A \cap B) = P(A|B)P(B))$$

$$= P(X_{n_2} = j \mid X_0 = i)\, P(X_{n_1} = i)$$

$$= (P^{n_2}_{ij})(\alpha P^{n_1})_i.$$

Similarly,

$$P(X_{n_1} = i,\ X_{n_1+n_2} = j,\ X_{n_1+n_2+n_3} = k) = (\alpha P^{n_1})_i\ P^{n_2}_{ij}\ P^{n_3}_{jk}$$

Proof Observe that

$$P(X_{n_1} = i,\ X_{n_1+n_2} = j,\ X_{n_1+n_2+n_3} = k)$$

$$= P(X_{n_1+n_2+n_3} = k \mid X_{n_1} = i,\ X_{n_1+n_2} = j)\, P(X_{n_1} = i,\ X_{n_1+n_2} = j)$$

$$= P(X_{n_1+n_2+n_3} = k \mid X_{n_1+n_2} = j)\, P(X_{n_1} = i,\ X_{n_1+n_2} = j)$$

$$= P^{n_3}_{jk}\ P^{n_2}_{ij}\ (\alpha P^{n_1})_i$$

Problem 1. Consider a Markov chain modeling a climate with three possible weather states: clear, rain, and snow. The day to day transition probabilities between states are given by

$$P = \begin{array}{c c} & \begin{array}{ccc} c & r & s \end{array} \\ \begin{array}{c} c \\ r \\ s \end{array} & \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \end{array}.$$

We also suppose that the initial distribution is $\alpha = (0.2, 0.3, 0.5)$. Use R to compute the following probabilities. Before doing so, convert the description into random variable notation, and then matrix notation.

a. The probability it's raining 5 days from now, given that it's snowing today.

b. The probability it's raining 5 days from now.

c. The probability it's raining 5 days from now and snowing 10 days from now, given that it's raining today.

d. The probability it's raining 5 days from now, snowing 10 days, and is clear 12 days from now, given that it's raining today.

e. The probability it's clear today, raining 5 days from now, snowing 10 days, and is clear 12 days from now.

f. The long term probabilities that it's clear, raining, or snowing. *Try computing $P^{10}, P^{20}, P^{30}, P^{40}$ using R.*

1. $P(X_5 = r \mid X_0 = s) = P^5_{sr}$

2. $P(X_5 = r) = (\alpha P^5)_r$

3. $P(X_{10} = s, X_5 = r \mid X_0 = r)$

$= P(X_{10} = s \mid X_5 = r, X_0 = r) P(X_5 = r \mid X_0 = r)$

$= P^5_{rs} P^5_{rr}$

4. $P(X_{12} = c, X_{10} = s, X_5 = s \mid X_0 = r)$

$= P(X_{12} = c \mid X_{10} = s, X_5 = s, X_0 = r) P(X_{10} = s, X_5 = s \mid X_0 = r)$

$= P(X_{12} = c \mid X_{10} = s) P(X_{10} = s \mid X_5 = s, X_0 = r) P(X_5 = s \mid X_0 = r)$

$= P^2_{sc} P^5_{ss} P^5_{rs}$

5. $P(X_{12} = c, X_{10} = s, X_5 = r, X_0 = c)$

$= P^2_{sc} P^5_{rs} P^5_{cr} \alpha_c$

6. look at row of large power like P^{50}

(should see $2/11$, $3/11$, $6/11$)

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```{r}
alpha = c(0.2, 0.3, 0.5)
P = matrix(c(0.1, 0.3, 0.6,
             0, 0.4, 0.6,
             0.3, 0.2, 0.5), nrow = 3, ncol = 3, byrow = T)

(P %^% 5)[3,2]
(alpha %*% (P %^% 5))[2]
(P %^% 5)[2,3] * (P %^% 5)[2,2]
(P %^% 2)[3,1] * (P %^% 5)[3,3] * (P %^% 5)[2,3]
(P %^% 2)[3,1] * (P %^% 5)[2,3] * (P %^% 5)[1,2] * alpha[1]
(P %^% 50)[1,]
```
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```
[1] 0.27272
[1] 0.272728
[1] 0.1487688
[1] 0.05355381
[1] 0.005355479
[1] 0.1818182 0.2727273 0.5454545
```

Problem 2. Let's go back to the random walk on a graph using the graph from last time. Use R to compute the matrices $P^5, P^{25}, P^{50}, P^{75}$.

a. Suppose the random walk has been run for a large but finite number of steps. Which vertex is the most likely state for the random walk to be in at the end? Least likely?

b. What do you notice about the relationship between the long term probabilities and the degrees of each vertex? Note that the degree of a vertex is the number of neighbors it has.

c. Assume that the random walk starts at vertex 1. What is the distribution of the walker's location after 5 steps?

d. Does your answer to the previous question change if we assume that the random walk starts at vertex 4?

e. You probably noticed the rows of P^{75} are all identical. What is the practical meaning of this in terms of the initial state of the random walk?

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```{r}
P = matrix(c(0, 1/3, 0, 1/3, 1/3, 0,
             1/2, 0, 0, 1/2, 0, 0,
             0, 0, 0, 1/2, 1/2, 0,
             1/4, 1/4, 1/4, 0, 1/4, 0,
             1/4, 0, 1/4, 1/4, 0, 1/4,
             0, 0, 0, 0, 1, 0), nrow = 6, ncol = 6, byrow = T)
P %^% 5
P %^% 25
P %^% 50
P %^% 75
```
````

```
          [,1]      [,2]       [,3]      [,4]      [,5]       [,6]
[1,] 0.1588542 0.1425058 0.10590278 0.2484086 0.3022280 0.04210069
[2,] 0.2137587 0.1085069 0.12999132 0.2697483 0.2035590 0.07443576
[3,] 0.1588542 0.1299913 0.10026042 0.2615017 0.3044705 0.04492188
[4,] 0.1863064 0.1348741 0.13075087 0.2343750 0.2540148 0.05967882
[5,] 0.2266710 0.1017795 0.15223524 0.2540148 0.1727431 0.09255642
[6,] 0.1263021 0.1488715 0.08984375 0.2387153 0.3702257 0.02604167
          [,1]      [,2]      [,3]      [,4]      [,5]       [,6]
[1,] 0.1874715 0.1250147 0.1249818 0.2499965 0.2500549 0.06248060
[2,] 0.1875220 0.1249887 0.1250140 0.2500027 0.2499576 0.06251497
[3,] 0.1874727 0.1250140 0.1249826 0.2499967 0.2500525 0.06248147
[4,] 0.1874974 0.1250014 0.1249983 0.2499997 0.2500050 0.06249822
[5,] 0.1875412 0.1249788 0.1250262 0.2500050 0.2499207 0.06252801
[6,] 0.1874418 0.1250299 0.1249629 0.2499929 0.2501120 0.06246042
       [,1]  [,2]  [,3] [,4] [,5]       [,6]
[1,] 0.1875 0.125 0.125 0.25 0.25 0.06250000
[2,] 0.1875 0.125 0.125 0.25 0.25 0.06250000
[3,] 0.1875 0.125 0.125 0.25 0.25 0.06250000
[4,] 0.1875 0.125 0.125 0.25 0.25 0.06250000
[5,] 0.1875 0.125 0.125 0.25 0.25 0.06250000
[6,] 0.1875 0.125 0.125 0.25 0.25 0.06250001
       [,1]  [,2]  [,3] [,4] [,5]   [,6]
[1,] 0.1875 0.125 0.125 0.25 0.25 0.0625
[2,] 0.1875 0.125 0.125 0.25 0.25 0.0625
[3,] 0.1875 0.125 0.125 0.25 0.25 0.0625
[4,] 0.1875 0.125 0.125 0.25 0.25 0.0625
[5,] 0.1875 0.125 0.125 0.25 0.25 0.0625
[6,] 0.1875 0.125 0.125 0.25 0.25 0.0625
```