§ 3.1 Limiting Distributions

Def A probability vector $\lambda = (\lambda_1, \dots, \lambda_m)$ is called a limiting distribution for a Markov chain $(X_n)_{n\geq 0}$ with transition matrix P if for each $j = 1, \dots, m$

$$\lim_{n\to\infty} P^n_{ij} = \lambda_j \quad \text{for any } i = 1, \dots, m.$$

Equivalently, λ is a limiting distribution if

$$\lim_{n\to\infty} P^n = \Lambda = \begin{bmatrix} - \lambda - \\ - \lambda - \\ \vdots \\ - \lambda - \end{bmatrix}$$

or if $\lim_{n\to\infty} \alpha P^n = \lambda$ for any initial distribution α.

Goals

1. Do all Markov chains have limiting distributions? Which do?
2. How do we compute limiting distributions? (analytically, not just numerically)
3. understand relationship between limiting distributions and time

Example (2-state Markov chain) Suppose $0 \le p, q \le 1$, $p+q \ne 0, 1$ and consider the Markov chain with transition diagram

$$\begin{tikzcd} 1 \arrow[loop left, "1-p"] \arrow[r, bend left, "p"] & 2 \arrow[loop right, "1-q"] \arrow[l, bend left, "q"] \end{tikzcd}$$

Then $$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}.$$ We can diagonalize P

That is, P has eigenvalues 1 and $1-p-q$ and eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -p \\ q \end{bmatrix}$ so that

$P = VDV^{-1}$ where $V = \begin{bmatrix} 1 & -p \\ 1 & q \end{bmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 1-p-q \end{pmatrix}$

That means
$$\begin{aligned} P^n &= (VDV^{-1})(VDV^{-1})\cdots(VDV^{-1}) \\ &= VD^nV^{-1} \\ &= V\begin{pmatrix} 1 & 0 \\ 0 & (1-p-q)^n \end{pmatrix} V^{-1} \\ &\to V\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V^{-1} = \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{pmatrix} \end{aligned}$$

And so $\lambda = \left(\frac{q}{p+q}, \frac{p}{p+q} \right)$ is the limiting distribution for this Markov chain.

<u>Theorem</u> Let $(X_n)_{n \geq 0}$ be a Markov chain with limiting distribution λ. Then the long-term expected proportion of time the Markov chain spends in state j is given by λ_j for each $j \in S$.

<u>Proof</u> Let $j \in S$ and define for each $k \geq 0$

$$I_k = \begin{cases} 1 & \text{if } X_k = j \\ 0 & \text{if } X_k \neq j. \end{cases}$$

Suppose $X_0 = i$. Then the long-term expected proportion of time spent at j is

$$\begin{aligned} \lim_{n \to \infty} E\left[\frac{1}{n} \sum_{k=0}^{n-1} I_k \,\middle|\, X_0 = i \right] &= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E\left[I_k \mid X_0 = i \right] \\ &= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P(X_k = j \mid X_0 = i) \\ &= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P_{ij}^k \\ &\overset{(*)}{=} \lim_{n \to \infty} P_{ij}^n = \lambda_j \end{aligned}$$

(*) Note if $\lim_{n \to \infty} a_n = L$, then $\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = L$

Problem 1. Consider the Markov chain with transition matrix

$$P = \begin{bmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{bmatrix}.$$

Use the formula for the limiting distribution of a 2-state Markov chain to find $\lim_{n\to\infty} P^n$ and verify your answer with R by computing P^{40}, P^{50}, P^{60}.

$p = 3/4$, $q = 2/3$, so limiting distribution is $\left(\frac{2/3}{3/4+2/3}, \frac{3/4}{3/4+2/3}\right)$

$= (8/17, 9/17)$ and $\lim_{n\to\infty} P^n = \begin{pmatrix} 8/17 & 9/17 \\ 8/17 & 9/17 \end{pmatrix}$

Problem 2. Consider the Markov chain with transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 3/4 \\ 0 & 2/3 & 1/3 \end{bmatrix}.$$

a. Draw the transition state diagram for this Markov chain.

b. Does this Markov chain have a limiting *matrix*? That is, does $\lim_{n\to\infty} P^n$ exist? If so, compute it without using R, and then verify your answer with R. *Hint: notice this is a block-diagonal matrix.*

c. Does this Markov chain have a limiting *distribution*?

d. We will soon use the term *communication class* to describe a set of states. What do you think this term means? How many communication classes do you think this Markov chain has?

(a)

(b) Yes, it's $\lim_{n\to\infty} P^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8/17 & 9/17 \\ 0 & 8/17 & 9/17 \end{pmatrix} = \Lambda$

(c) No, there is a dependence between initial state and the long-term behavior of the chain. More concretely, the rows of the limiting matrix Λ are not all identical.

(d) 2 communication classes: $\{1\}$ and $\{2,3\}$

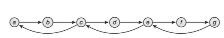
Problem 3. A *6-cycle* is a graph with 6 vertices that are arranged in a circle so that each vertex has two neighbors. Consider the random walk on a 6-cycle. Its transition matrix is given by

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}.$$

Does $\lim_{n\to\infty} P^n$ exist? Why or why not? Does this Markov chain have a limiting distribution?

No, $\lim_{n\to\infty} P^n$ does not exist. Notice $\lim_{n\to\infty} P^{2n}$ and $\lim_{n\to\infty} P^{2n+1}$ are different, so the long-term behavior of P^n oscillates on even and odd steps. This is because, for example, transitions from a state back to itself can only happen after an even number of steps and transitions from a state to its neighbors can only happen on an odd-numbered step.

Problem 4. Consider the Markov chain with transition state diagram below. Probabilities are missing in the diagram, but let's assume that for states c and e the probabilities for the two eminating arrows are each $1/2$. Using R, what do you notice about powers of P? Does this Markov chain have a limiting *matrix*? Does this Markov chain have a limiting *distribution*? We will soon use the term *periodicity of a state*. What do you think it means? What do you think is the periodicity of each state in this chain.



No limiting matrix. Periodicity of each state is 3.

Problem 5. Make a conjecture about two necessary conditions for a Markov chain to have a limiting distribution. Use the terms *communication class* and *periodicity*.

Necessary conditions to have a limiting distribution:

one communication class, states have periodicity 1.